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CALCULATION OF THE TURBULENT BOUNDARY LAYER AND OF TRANSITION ON A PLANE PLATE

by R. Hirsch

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ERRATA

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February 1968

1. Page 1, line 11; Instead of "For a critical Reynolds number of 0.0037", insert "For a critical Reynolds number

$$\mathcal{R}_c = 0{,}0037 \frac{U_0^2 \lambda^2}{\nu^2};"$$

2. Page 1, line 14: Instead of "over the segment of $\frac{1}{2}\mathcal{R}_c$ " insert "over the segment $\frac{1}{2}\mathcal{R}_c, \mathcal{R}_c$. "

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CALCULATION OF THE TURBULENT BOUNDARY LAYER AND OF TRANSITION ON A PLANE PLATE

By R. Hirsch

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CALCULATION OF THE TURBULENT BOUNDARY LAYER AND OF TRANSITION ON A PLANE PLATE /1*

R.Hirsch

ABSTRACT. Calculations of the turbulent boundary layer and transition on a plane plate are derived in detail, including the following: study of a solution of two Navier-Stokes equations inducing a stationary field analogous to the average turbulent field; existence of an asymptotic solution and determination of the development of friction; study of the effect of an exterior harmonic perturbation of wavelength λ. *For a critical Reynolds number of 0.0037, the Blasius state is unable to exist and must be replaced by the newly developed state. The perturbations permitting such substitu-*tion and formed over the segment of $\frac{1}{2}\,\mathfrak{R}_c$ are derived, and the calculations are extrapolated to the compressible state with heat exchange. Comparisons of the results with practical experiments show agreement with the Blasius theory of the laminar regime. A possibility exists to maintain the laminar state by replacing the rigid wall with an elastic membrane whose tension would be made dependent on the pulsation of the exterior perturbation which ordinarily causes passage to the turbulent state.

INTRODUCTION

The classical theories of the turbulent boundary layer attribute to the constituent particles the property of passing from one level to the other and, by the process of collision, to lose or gain momentum, to the profit or detriment of neighboring particles.

This leads to the concept of mixing lengths of the order of a millimeter. The process reduces to putting in force, at the macroscopic scale, the data which at the infinitely small scale of the kinetic theory of gases explain the existence of viscosity stresses. However, in the kinetic theory of gases it is a question of individualized molecules mutually isolated without interposition of a continuous medium since the lengths of the mean free paths are so small that, at the macroscopic scale, the total aggregate can be regarded as being continuous, at least in the statistical sense.

One could ask here why the same basic phenomenon is encountered at two such widely differing scales, and it is not easy to conceive how the sensible reality

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* Numbers in the margin indicate pagination in the foreign text.

* - See Errata

of air at atmospheric pressure could, at the scale of a millimeter, be compatible with the concept of a discontinuous medium which presupposes the mixing of particles from layer to layer.

Nevertheless, is it really impossible, within the framework of general equations of continuous media whose rational base is fairly solid, to cause the appearance of the essential properties of fluid motion in turbulent boundary layers? This is finally the question that we are attempting to answer, although by frequently cumbersome means.

For this reason, it is necessary to give a proper elucidation of all considered points.

For example, why would the two approaches to the problem of boundary layers, where one is constituted by the Blasius theory (with its hypotheses of invariance of pressure) and the other by the method proposed here, be the object to two types of flow existing in nature effectively and successively?

No doubt, we can invoke here the notion of maximum energy dissipation to justify the preferential establishment of laminar flow (at Reynolds numbers below 1000, the laminar friction is actually greater than the turbulent friction). However, this merely represents an overall explanation which does not furnish complete information.

Why does an abrupt condensation of rotation take place in a very thin sublayer, as soon as the turbulent state is established? The equations confirm this phenomenon and experiments substantiate it; however, no meaningful explanation exists at present.

Still other questions remain in suspense.

No matter how this might be, it will be demonstrated below that it is /2
possible, in a continuous medium, to demonstrate theoretically a state differing from the Blasius state, such that the velocity field, the evolution of boundary layer thickness, the evolution of friction, and the state of turbulence will be more or less in agreement with the data obtained from practical experiments on the turbulent boundary layer of a plane plate. It is also possible to investigate the connectivity zone with the Blasius state, i.e., the zone of transition, and thus to obtain determinations which satisfactorily agree with the experimental results. Finally, a study of the compressible case with heat exchange can be undertaken. This constitutes the general justification of our essay.

We wish to express thanks to all those who aided our work with advice and comprehension, Prof. Eichelbrenner, Prof. Oudart, and Chief Engineer Vernotte. The paper is dedicated to the memory of my teacher Prof. Albert Toussaint.

PART I

STUDY OF THE INCOMPRESSIBLE STATIONARY CASE

CHAPTER I /5

FUNDAMENTAL FORMULAS AND STUDY OF THE ACTUAL BOUNDARY LAYER

1. Fundamental Formulas

Let there be an incompressible two-dimensional flow in contact with the rectilinear wall of a plane plate; let U_0 be the value of the potential velocity on an abscissa segment x. The axes are fixed with respect to the wall, with the origin of x represented by the leading edge of the plate.

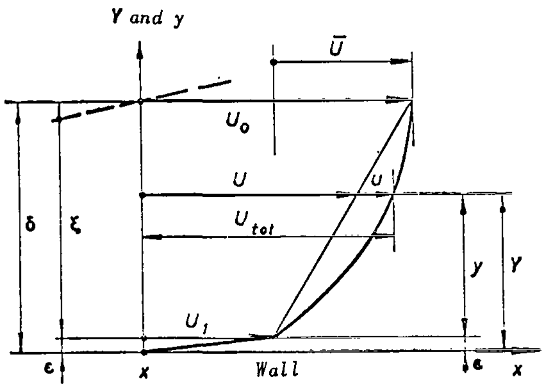

Fig.1

Let us take as basic diagram of first approximation of the velocity U, starting from the wall, a scheme showing the following (Fig.1):

1) a boundary sublayer ε, at whose top the velocity is U_1*; ε is extremely small so that $\vec{U}_1$ is more or less parallel to the wall;
2) a boundary layer ξ known as "actual boundary layer" in which the velocity increases from U_1 to a value extremely close to U_0.

It is stipulated that U_0 is constant and parallel to the wall. At some /6
level Y (where $y = Y - \epsilon$), comprised between ε and $(\xi + \epsilon)$, we will assume that the tangential velocity U principally follows a linear distribution in y between U_1 and U_0, such that

$$U(y) = U + \overline{U}\frac{y}{\xi} \qquad \text{where} \qquad \overline{U} = U_0 - U_1.$$

The normal velocity will be denoted by V(y).

For the time being, we will use ξ for a selected arbitrary thickness, with the distribution U(y) linear in y, so as to represent the mean field of a turbulent boundary layer in the best possible manner.

It will be assumed that the derivatives with respect to x, U'_{1_x}, $\overline{U}'_x = -U'_{1_x}$ are very small relative to the partial derivative of U with respect to y; it will

* Between the wall and the border line of the sublayer, it is necessary, so as to have zero velocity U at the wall, to assume the presence of particles in rotation rolling along the wall at the mean driving velocity $\frac{U_1}{2}$.

also be assumed that ξ'_x, ϵ'_x are of the same order of magnitude as U'_{1_x}.

Our investigation primarily is concerned with the problem in y, in some section x of the actual boundary layer ξ, comprising a determination of the laws u(y), v(y) of the complementary velocities assumed as small relative to U(y), which permits satisfying the Navier-Stokes equations at the condition of continuity and at the boundary conditions to be respected.

In this problem, ϵ, ξ, U_1 (and thus $\overline{U}$) will be considered as given.

This yields a velocity distribution of the boundary layer, which - for convenience - will be denoted by the term "stationary turbulent" boundary layer. The justification of this designation will appear only in a nonstationary study.

The next problem to be attacked will be an investigation of the evolution of $\overline{U}$, U_1, ϵ, ξ as a function of x and a study of the connectivity conditions with the laminar boundary layer which (with respect to x) precedes the "turbulent" layer.

2. General Equations

Our computational hypotheses, referring to the orders of magnitude, will be the following:

U_0, $\overline{U}$ (thus, $U_1 = U_0 - \overline{U}$), principal ξ as well as $\overline{U}\frac{y}{\xi}$;

u and $\frac{\partial u}{\partial y}$ of the first order of smallness;

v and all derivatives with respect to x, of the second order.

The continuity condition, applied to the velocity components U and V of the basic diagram $\left(U = U_1 + \overline{U}\frac{y}{\xi}\right)$, leads to

$$\frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x} = -\left[y\left(\frac{\overline{U}}{\xi}\right)'_x + U'_{1_x}\right],$$

whence* /7

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* It should be noted here that the streamline passing through $y = \xi$ has the following slope, even if $U'_{1_x} = 0$:

$$[y'_x]_\xi = \left[\frac{V}{U}\right]_\xi = -\frac{1}{U_0}\left(\frac{\overline{U}}{\xi}\right)'_x \cdot \frac{\xi^2}{2} = \frac{1}{2U_0}(\overline{U}\xi'_x - \xi\overline{U}'_x).$$

If $U'_x \simeq 0$, we have $\frac{\overline{U}}{2U_0}\xi''_x$ as remainder and thus $\frac{[y'_x]\xi}{\xi'_x} = \frac{\overline{U}}{2U_0}$. This shows that, in general, the normal velocity at the boundary ξ is not zero and that the stream traverses this boundary.

$$V = -\left[\left(\frac{\overline{U}}{\xi}\right)''_x \cdot \frac{y^2}{2} + U'_{1_x} y\right] + \text{const}.$$

Since, for $y = 0$, $V \cong 0$, a plane plate is involved, the constant will be zero. Then, the following expressions are obtained:

$$\frac{\partial U}{\partial y} = \frac{\overline{U}}{\xi}, \quad \frac{\partial U}{\partial x} = y\left(\frac{\overline{U}}{\xi}\right)'_x, \quad \frac{\partial^2 U}{\partial y^2} = 0, \quad \frac{\partial^2 U}{\partial x^2} = y\left(\frac{\overline{U}}{\xi}\right)''_{x^2};$$

$$\frac{\partial V}{\partial y} = -y\left(\frac{\overline{U}}{\xi}\right)'_x - U'_{1_x}, \quad \frac{\partial V}{\partial x} = -y\left(\frac{\overline{U}}{\xi}\right)''_{x^2} - U''_{1_{x^2}}, \quad \frac{\partial^2 V}{\partial y^2} = -\left(\frac{\overline{U}}{\xi}\right)'_x,$$

$$\frac{\partial^2 V}{\partial x^2} = -y\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} - U'''_{1_{x^3}}.$$

Finally, the rotations are written in the form

$$\Omega = \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) = -\frac{1}{2}\left[y\left(\frac{\overline{U}}{\xi}\right)''_{x^2} + U''_{1_{x^2}} + \frac{\overline{U}}{\xi}\right] \cong -\frac{1}{2} \cdot \frac{\overline{U}}{\xi}.$$

For the first members of the Navier-Stokes equations, we thus obtain

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial x} = \left(\overline{U}\frac{y}{\xi} + U_1\right)\left(\frac{\overline{U}}{\xi}\right)'_x \cdot y - \left\{\left(\frac{\overline{U}}{\xi}\right)'_x \frac{y^2}{2} + U'_{1_x} \cdot y\right\}\frac{\overline{U}}{\xi} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial x},$$

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial y} = -\left(\overline{U}\frac{y}{\xi} + U_1\right) y \cdot \left(\frac{\overline{U}}{\xi}\right)''_{x^2}$$

$$+\left\{\left(\frac{\overline{U}}{\xi}\right)'_x \cdot \frac{y^2}{2} + U'_{1_x} \cdot y\right\}\left\{y\left(\frac{\overline{U}}{\xi}\right)'_x + U'_{1_x}\right\} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial y}.$$

For the second members, it follows that

$$\nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = \nu \cdot y\left(\frac{\overline{U}}{\xi}\right)''_{x^2},$$

$$\nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) = -\nu\left[y\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} + U'''_{1_{x^3}} + \left(\frac{\overline{U}}{\xi}\right)'_x\right],$$

where ν is the kinematic viscosity, p the pressure, and ρ the specific mass.

The term $V\dfrac{\partial V}{\partial y}$ contains products of derivatives with respect to x; this term is negligible in view of our convention regarding the orders of magnitude. /8

Such a distribution scheme of the velocities U (at constant gradient U'_y), if it satisfies the continuity condition, obviously does not satisfy the Navier-Stokes equations.

To the preceding velocities U, V it is necessary to add unknown perturbation components u, v deriving from a stream function ψ (in which case the continuity condition will be satisfied) by means of which the Navier equations relative to the total velocities $U_{tot} = U + u$, $V_{tot} = V + v$ will be satisfied.

It is possible to give a solution only in the case in which $\frac{u}{U}$ $\left(\text{and } \frac{v}{U}\right)$ can be assumed as small (and very small), a case where a linearization method similar to the Schlichting perturbation method will be applicable.

We will know only afterwards whether the admitted simplification had been justified*.

2.1 Complementary Stream Function

Let us now consider a stream function $\psi(x, y)$ such that $\psi'_y = u$, $-\psi'_x = v$ and let us derive the Navier equations with the total velocity components:

$$U_{tot} = \frac{\overline{U}}{\xi} y + U_1 + \psi'_y, \qquad V_{tot} = -\left[\frac{y^2}{2}\left(\frac{\overline{U}}{\xi}\right)'_x + U'_{1_x} \cdot y + \psi'_x\right].$$

Since $\frac{\partial U_{tot}}{\partial x} = \left(\frac{\overline{U}}{\xi}\right)'_x y + U'_{1_x} + \psi''_{yx} = -\frac{\partial V_{tot}}{\partial y}$, they are, in complete form,

$$\left[\frac{\overline{U}}{\xi} y + U_1 + \psi'_y\right]\left[\left(\frac{\overline{U}}{\xi}\right)'_x y + U'_{1_x} + \psi''_{yx}\right] - \left[\left(\frac{\overline{U}}{\xi}\right)'_x \frac{y^2}{2} + U'_{1_x} \cdot y + \psi'_x\right]\left[\frac{\overline{U}}{\xi} + \psi''_{y^2}\right]$$
$$+ \frac{1}{\rho} \cdot \frac{\partial p}{\partial x} = \nu\left[y\left(\frac{\overline{U}}{\xi}\right)''_{x^2} + U''_{1_{x^2}} + \psi'''_{yx^2} + \psi'''_{y^3}\right]$$
$$-\left[\frac{\overline{U}}{\xi} y + U_1 + \psi'_y\right]\left[\left(\frac{\overline{U}}{\xi}\right)''_{x^2} \cdot \frac{y^2}{2} + U''_{1_x} \cdot y + \psi''_{x^2}\right] + \left[\left(\frac{\overline{U}}{\xi}\right)'_x \cdot \frac{y^2}{2} + U'_{1_x} \cdot y + \psi'_x\right]$$
$$\times \left[\left(\frac{\overline{U}}{\xi}\right)'_x y + U'_{1_x} + \psi''_{xy}\right]$$
$$+ \frac{1}{\rho} \cdot \frac{\partial p}{\partial y} = -\nu\left[\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} \cdot \frac{y^2}{2} + U'''_{1_{x^3}} \cdot y + \psi'''_{x^3} + \left(\frac{\overline{U}}{\xi}\right)'_x + \psi'''_{xy^2}\right].$$

The pressure p can be eliminated by deriving the first of these expressions with respect to y and the second with respect to x and then subtracting term /9
by term. This process is that used by Schlichting which, classically, leads to what we commonly call the fundamental relation of formation of rotations:

* In fact, it will be found that, at the maximum, $\frac{u}{U}$ assumes a value of 0.17 which is not completely satisfying but which also does not contradict the initial hypothesis.

$$\frac{d\,\Omega_{tot}}{dt} = \nu \cdot \Delta^2 \Omega_{tot}.$$

Here, we obtain

$$\left\{\left(\frac{\overline{U}}{\xi} + \psi''_{y^2}\right)\left\{\left[\left(\frac{\overline{U}}{\xi}\right)'_x y + U'_{1_x} + \psi''_{xy}\right]\right\} + \left[\frac{\overline{U}}{\xi} y + U_1 + \psi'_y\right]\left[\left(\frac{\overline{U}}{\xi}\right)'_x + \psi'''_{y^2x}\right]$$

$$-\left(\left\{\left(\frac{\overline{U}}{\xi}\right)_x y + U'_{1_x} + \psi''_{xy}\left\{\left[\frac{\overline{U}}{\xi} + \psi''_{y^2}\right] + \left[\left(\frac{\overline{U}}{\xi}\right)'_x \frac{y^2}{2} + U'_{1_x} \cdot y + \psi'_x\right]\psi'''_{y^3}\right)\right\}$$

$$+\left\{\left(\left\{\left[\left(\frac{\overline{U}}{\xi}\right)'_x \cdot y + U'_{1_x} + \psi''_{yx}\right]\left[\left(\frac{\overline{U}}{\xi}\right)''_{x^2} \cdot \frac{y^2}{2} + U''_{1_{x^2}} \cdot y + \psi''_{x^2}\right]\right\}\right.$$

$$\left.+\left(\frac{\overline{U}}{\xi} y + U_1 + \psi'_y\right)\left[\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} \cdot \frac{y^2}{2} + U'''_{1_{x^3}} \cdot y + \psi'''_{x^3}\right]\right)$$

$$-\left(\left\{\left[\left(\frac{\overline{U}}{\xi}\right)''_{x^2} \cdot \frac{y^2}{2} + U''_{1_{x^2}} \cdot y + \psi''_{x^2}\right]\left[\left(\frac{\overline{U}}{\xi}\right)'_x \cdot y + U'_{1_x} + \psi''_{xy}\right]\right\}\right.$$

$$\left.+\left[\left(\frac{\overline{U}}{\xi}\right)'_x \cdot \frac{y^2}{2} + U'_{1_x} y + \psi'_x\right]\left[\left(\frac{\overline{U}}{\xi}\right)''_{x^2} y + U''_{1_{x^2}} + \psi'''_{x^2y}\right]\right)$$

$$= \nu\left[\right\}\psi''''_{y^2x^2} + \psi''''_{y^4} + \left(\frac{\overline{U}}{\xi}\right)''_{x^2}\left.\right\}$$

$$+\left\}\left(\frac{\overline{U}}{\xi}\right)''''_{x^4} \cdot \frac{y^2}{2} + U''''_{1_{x^4}} \cdot y + \psi''''_{x^4} + \left(\frac{\overline{U}}{\xi}\right)''_{x^2} + \psi''''_{y^2x^2}\left\{\right]$$

which can be simplified (terms between $\}\}$).

According to our convention, the terms $\frac{\overline{U}}{\xi}\, y + U_1$ are principal terms. Those in ψ are small and of the first order. We will retain all terms up to the second order inclusive, by assuming ψ'_y to be of the first order and all derivatives with respect to x of the second order. This will eliminate the products in ψ'_x, $\psi^{(n)}_{y^{n-1}x}$ and $\psi^{(n)}_{y^{n-1}x} \cdot \left(\frac{U}{\xi}\right)'_x$, $U'_{1x}\left(\frac{U}{\xi}\right)'_x$, etc. which are at least of the third order.

This first yields

$$\left[\left(\frac{\overline{U}}{\xi}\right) y + U_1 + \psi'_y\right]\left[\left\{\left(\frac{\overline{U}}{\xi}\right)'_x + \psi'''_{y^2x}\right\} + \left\{\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} \cdot \frac{y^2}{2} + U'''_{1_{x^3}} \cdot y + \psi'''_{x^3}\right\}\right]$$

$$-\left[\left(\frac{\overline{U}}{\xi}\right)'_x \cdot \frac{y^2}{2} + U'_{1_x} y + \psi'_x\right]\left[\psi'''_{y^3} + \left\}\left(\frac{\overline{U}}{\xi}\right)''_{x^2} y + U''_{1_{x^2}} + \psi'''_{x^2y}\right\{\right]$$

$$= \nu\left[\left(\frac{\overline{U}}{\xi}\right)''''_{x^4} \cdot \frac{y^2}{2} + U''''_{1_{x^4}} \cdot y + 2\left(\frac{\overline{U}}{\xi}\right)''_{x^2} + \psi''''_{x^4} + 2\,\psi''''_{x^2y^2} + \psi''''_{y^4}\right].$$

On suppressing the products of derivatives in x formed with themselves and with the quantity ξ, we obtain /10

$$\left[\left(\frac{\overline{U}}{\xi}\right) y + U_1 + \psi'_y\right]\left[\psi'''_{y^2x} + \psi'''_{x^3} + \left(\frac{\overline{U}}{\xi}\right)'_x + \left(\frac{\overline{U}}{\xi}\right)'''_{x^3} \cdot \frac{y^2}{2} + U'''_{1_{x^3}} \cdot y\right]$$
$$-\left\{\left[\left(\frac{\overline{U}}{\xi}\right)'_x \cdot \frac{y^2}{2} + U'_{1_x}\right]\psi''_{x^2}\right\}$$
$$= \nu\left[\psi''''_{x^4} + 2\,\psi''''_{x^2y^2} + \psi''''_{y^4} + \left(\frac{\overline{U}}{\xi}\right)''''_{x^4} \cdot \frac{y^2}{2} + U''''_{1_{x^4}} \cdot y + 2\left(\frac{\overline{U}}{\xi}\right)''_{x^2}\right]$$

which is the equation of definition of the function ψ.

3. Minor Role of Viscosity in the Actual Boundary Layer; Reduced Form of the Fundamental Equation

The kinematic viscosity ν is such that $\frac{\nu}{U_{tot} \cdot \xi}$ is always very small* (with U_{tot} being the first bracket $\left[\left(\frac{\overline{U}}{\xi}\right) y + U_1 + \psi'_y\right]$).

It is logical to assign to this viscosity the same infinitesimal order as that of the derivatives with respect to x, namely the second order.

On the right-hand side, the term $\frac{1}{U_{tot}} \nu \cdot \psi''''_{y^4}$ is the principal term and also comes out as being of the third order, so that it can be neglected for $\xi \cdot U'''_{1_{x^3}}$, $\left(\frac{U_0 - U_1}{\xi}\right)'_x$, ψ'''_{y^2x}, on the left-hand side, all of which are of the second order.

Thus, at low viscosity, i.e. for relatively high Reynolds numbers (which will be the case in the usual problems), the form of the fundamental equation derived from the Navier equations is such that, at x = const (i.e., in each segment x), the solution expressed in y will not depend at all – or at most very little – on the viscosity.

This observation is valid only in the actual boundary layer where U_{tot} is of the same order of magnitude as U_0 ($U_1 < U_{tot} < U_0$).

This statement is no longer valid in the sublayer where, close to the wall, $U_{tot} \to 0$. This is the reason for the fact that neglecting of the terms in ν, at the present state of the problem, will not eliminate the influence of the vis-

* With respect to $U_1 \geqq 0.45\ U_0$ which fixes the order of magnitude to be retained to the first order for transition Reynolds numbers $\Re_c$ of 10^5 to 10^6, the term $\frac{\nu}{U_{tot}\ \xi}$ is below 2×10^{-3} to 0.6×10^{-3}.

cosity in the overall problem.

Thus, in the actual boundary layer, the equation defining ψ is reduced to

$$\psi'''_{xy^2} + \psi'''_{x^3} + \left(\frac{U}{\xi}\right)'_x + \frac{y^2}{2}\cdot\left(\frac{U}{\xi}\right)'''_{x^3} + y\,U'''_{1x^3} \cong 0.$$

3.1 Useful Form of the Equation of Definition of the Complementary Stream Function /11

The term $\psi(x, y)$ can also be resolved into a sum of products of functions φ_n of x and y and f_n of x alone, i.e., we can put

$$\psi = \sum_n \psi_n$$

with $\psi_n = \varphi_n(x, y) \cdot f_n(x)$, where ψ_n is small and of the first order while the number n of functions is assumed as finite.

Let us consider the quantities f_n as being small (and f'_{n_x} as very small), with the φ_n being of the principal order, φ'_{n_y} being principal and φ'_{n_x} being very small*. Since we have to retain only terms up to the second order in the equation in ψ, we will have

$$\psi'_x = \varphi'_x \cdot f + \varphi \cdot f'_x \cong \varphi \cdot f'_x, \qquad \psi''_{x^2} = \varphi''_{x^2} f + 2\,\varphi'_x f'_x + \varphi f''_{x^2} \cong \varphi f''_{x^2},$$
$$\psi'''_{x^3} = \varphi'''_{x^3} f + 3\,\varphi''_{x^2} f'_x + 3\,\varphi'_x f''_{x^2} + \varphi f'''_{x^3} \cong \varphi f'''_{x^3},$$
$$\psi'_y = \varphi'_y f, \qquad \psi''_{y^2} = \varphi''_{y^2} f, \qquad \psi''''_{y^4} = \varphi''''_{y^4} \cdot f$$
$$\text{and} \quad \psi'''_{y^2x} = \varphi'''_{y^2x} f + \varphi''_{y^2} f'_x \cong \varphi''_{y^2} \cdot f'_x.$$

The fundamental equation is written as

$$\sum_n \left[\varphi f'''_{x^3} + \varphi''_{y^2} f'_x\right] + \left(\frac{\bar{U}}{\xi}\right)'_x + \frac{y^2}{2}\cdot\left(\frac{\bar{U}}{\xi}\right)'''_{x^3} + y U'''_{1x^3} = 0.$$

This equation, which is of the second order in $\varphi(y)$, has the following general solution, with c, α, β, γ, φ_1, φ_2 being constants with respect to y:

---

* It will be shown later that φ_n cannot be considered as absolutely independent of x since it must satisfy $\left(\frac{\varphi''_y{}'}{\varphi}\right)_n = c^3 n$ where $c_n = \pi\,\frac{n}{\xi}$. This yields

$$\frac{\partial}{\partial x}\left(\frac{\varphi'_{y'}}{\varphi}\right) = -2\,\frac{n^2\pi^2}{\xi^3}\,\xi'_x$$

which is definitely of the infinitesimal second order, like ξ'_x.

$$\varphi = \varphi_1 e^{\overset{\circ}{i}cy} + \varphi_2 e^{-\overset{\circ}{i}cy} + \alpha + \beta y + \gamma y^2 \quad (\overset{\circ}{i} \text{ imaginary operator}).$$

Hence,

$$\varphi''_{y^2} = -c^2 (\varphi_1 e^{\overset{\circ}{i}cy} + \varphi_2 e^{-\overset{\circ}{i}cy}) + 2\gamma.$$

At each point x, the solution is such that (with α, β, γ being functions of x)

$$\sum_n [(-c^2 f'_x + f'''_{x^3})(\varphi_1 e^{\overset{\circ}{i}cy} + \varphi_2 e^{-\overset{\circ}{i}cy}) + f'''_{x^3}(\alpha + \beta y + \gamma y^2) + 2\gamma f'_x] + \left(\frac{\overline{U}}{\xi}\right)'_x + \frac{y^2}{2}\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} = 0.$$

3.2 Solution - Complementary Velocity Components /12

To satisfy this relation, it is necessary that the terms containing y in the various irreducible forms will be separately zero, i.e.*,

1) Terms in $\varphi_1 e^{\overset{\circ}{i}cy} + \varphi_2 e^{-\overset{\circ}{i}cy}$

$$f'''_{x^3} - c^2 f'_x = 0, \quad \text{whence} \quad f = f_1 e^{cx} + f_2 e^{-cx} + f_0$$

(where f_1, f_2, f_0 are constants).

This solution in f is valid only if $c'_x = 0$. We will demonstrate that, in reality, c'_x is small but not absolutely zero.

The above form will be a solution only in a relatively narrow domain Δ_x such that $\Delta_c = c'_x\Delta$ will be small with respect to c(x).

This leads to integrating the function f with respect to x, term by term, and thus to retain f_1, f_2, f_3 at their interior as constants that vary slowly from step to step.

2) Terms independent of y:

$$\sum_n \{\alpha f'''_{x^3} + 2\gamma f'_x\} + \left(\frac{\overline{U}}{\xi}\right)'_x = 0.$$

The solution has the form

$$\sum_n \{f'_x[\alpha c^2 + 2\gamma]\} + \left(\frac{\overline{U}}{\xi}\right)'_x = 0, \quad \text{with} \quad f'_x = c(f_1 e^{cx} - f_2 e^{-cx}).$$

* To each value n, values c_n, φ_{1n}, φ_{2n}, f_{1n}, etc. are associated. Except where necessary, we will omit the subscript n.

3) Terms in y:

$$U'''_{1x^3} + \sum_n \{ f'''_{x^3} \cdot \beta \} = 0.$$

4) Terms in y^2:

$$\sum_n \{ f'''_{x^3} \cdot \gamma \} + \frac{1}{2}\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} = 0.$$

The solution has the form

$$\sum_n c^2 \cdot f'_x \gamma + \frac{1}{2}\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} = 0.$$

Similarly,

$$\sum_n \{ c^2 f'_x \beta \} + U'''_{1x^3} = 0.$$

In front of $\left(\frac{\overline{U}}{\xi}\right)'_x$ and $\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}$ let us introduce coefficeents λ_n invariant /13 with respect to x such as $\sum_n \lambda_n = 1$, so as to permit substituting the two preceding relations containing the sign $\sum_n$ by 2n relations where this sign is no longer present. These are

$$f'_x(\alpha c^2 + 2\gamma) + \lambda\left(\frac{\overline{U}}{\xi}\right)'_x = 0,$$

$$c^2 f'_x \gamma + \frac{1}{2}\lambda\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} = 0$$

and

$$c^2 f'_x \beta = -\lambda U'''_{1x^3}.$$

Multiplying the first by $-c^2\gamma$, the second by $\alpha c^2 + 2\gamma$, and adding, we obtain

$$-c^2\gamma\lambda\left(\frac{\overline{U}}{\xi}\right)'_x + (\alpha c^2 + 2\gamma)\frac{\lambda}{2}\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} = 0,$$

whence

$$\alpha = -\frac{2\gamma}{c^2} + 2\gamma\frac{\left(\frac{\overline{U}}{\xi}\right)'_x}{\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}} = 2\gamma\left[\frac{\left(\frac{\overline{U}}{\xi}\right)'_x}{\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}} - \frac{1}{c^2}\right].$$

With

$$\gamma = -\frac{\lambda}{2\,c^3} \cdot \frac{\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}}{f_1\, e^{cx} - f_2\, e^{-cx}}$$

it follows that

$$\alpha = -\frac{1}{c^3} \cdot \frac{\lambda}{f_1\, e^{cx} - f_2\, e^{-cx}} \left[\left(\frac{\overline{U}}{\xi}\right)'_x - \frac{1}{c^2} \left(\frac{\overline{U}}{\xi}\right)'''_{x^3} \right].$$

These relations can also be written in the form

$$\beta\, f'_x = -\frac{\lambda}{c^2}\, U'''_{1x^3}$$

$$f'_x\, \gamma = -\frac{\lambda}{2\,c^2} \left(\frac{\overline{U}}{\xi}\right)'''_{x^3}, \qquad \alpha\, f'_x = -\frac{\lambda}{c^2} \left[\left(\frac{\overline{U}}{\xi}\right)'_x - \frac{1}{c^2} \left(\frac{\overline{U}}{\xi}\right)'''_{x^3} \right].$$

They show that $f'_x \cdot \gamma$ and $f'_x \cdot \alpha$ are small like $\left(\frac{\overline{U}}{\xi}\right)'_x$ and $\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}$.

From the first equations, at the interior of a step of an integration by parts (with γ and c being constants), we derive

$$\gamma\, \Delta f = -\frac{\lambda}{2\,c^2} \left(\frac{\overline{U}}{\xi}\right)'''_{x^3} \Delta x = -\frac{\lambda}{2\,c^2} \Delta \left(\frac{\overline{U}}{\xi}\right)''_{x^2},$$

which can be written as /14

$$\gamma\, f = -\frac{\lambda}{2\,c^2} \left(\frac{\overline{U}}{\xi}\right)''_{x^2}.$$

Similarly,

$$\beta\, f = -\frac{\lambda}{c^2}\, U''_{1x^2}.$$

(Since the integral relation is satisfied at the origin of the step, there is no need to introduce a complementary constant of integration.)

The expressions to be used with respect to the complementary velocity components thus will be, in each segment x,

$$u(y) = \sum_n (\varphi'_y\, f)_n = \sum_n \left(\{ f_1\, e^{cx} + f_2\, e^{-cx} + f_0 \} \left[\mathring{i}c\, (\varphi_1\, e^{\mathring{i}cy} - \varphi_2\, e^{-\mathring{i}cy}) + \beta + 2\,\gamma\, y \right] \right)_n$$

[with $f_n = (f_1\, e^{cx} + f_2\, e^{-cx} + f_0)_n$].

$$v(y) = -\sum_n (\varphi f'_x)_n = -\sum_n \left(\{ c(f_1 e^{cx} - f_2 e^{-cx}) \} [(\varphi_1 e^{\mathring{i}cy} + \varphi_2 e^{-\mathring{i}cy}) + \beta y + \alpha + \gamma y^2] \right)_n.$$

4. Study of the Boundary Conditions

For the problem in y, these conditions must be expressed at the borders of the actual boundary layer, with the segment x considered as being arbitrary.

It is first necessary that $u = 0$, for $y = 0$ (lower border with the sub-layer). From this, it follows that.

$$\varphi_1 - \varphi_2 = 0 \qquad \text{or} \qquad \varphi_1 = \varphi_2 = \varphi_0,$$

whence

$$u(y) = \sum_n (\{ f_1 e^{cx} + f_2 e^{-cx} + f_0 \} [-2\varphi_0 c \sin cy + \beta + 2\gamma y])_n$$

and

$$u(\xi) = \sum_n (\{ f_1 e^{cx} + f_2 e^{-cx} + f_0 \} [-2\varphi_0 c \sin c\xi + \beta + 2\gamma\xi])_n.$$

The term $u(\xi)$ must be zero if one takes into consideration that the thickness of the actual boundary layer is well defined by ξ. A priori, we know nothing in this respect and therefore must assume that $u(\xi)$ is very small but not necessarily zero. Since βf, γf are very small [since they are proportional to $\left(\frac{\bar{U}}{\xi}\right)''_x$, and $U'''_{1_{x^3}}$ which are assumed as very small], then also $f \sin c\xi$ is very small. It would have to be even absolutely zero if, in the fundamental equation in ψ, the terms in $U'''_{1_{x^3}}\left(\frac{\bar{U}}{\xi}\right)'''_{x^3}$ and $\left(\frac{\bar{U}}{\xi}\right)'_x$ were neglected. In that case, γ as well as β would be negligible and, to obtain $u(\xi) = 0$, it would be necessary that $\sin c_n\xi = 0$, i.e., $c_n = \frac{n\pi}{\xi}$.

Let us thus pose this condition and let us return to the expression containing $\left(\frac{\bar{U}}{\xi}\right)'''_{x^3}$ and $\left(\frac{\bar{U}}{\xi}\right)'_x$, in short $U'''_{1_{x^3}}$. /15

4.1 Residual Component u at the Upper Boundary

For $y = \xi$, the residual component of u, namely $\Delta u(\xi)$, will be

$$\Delta u(\xi) = \sum_n (f_n 2\gamma_n \cdot \xi + \beta_n) = -\sum_n \frac{\lambda_n}{n^2\pi^2} \xi^3 \left(\frac{\bar{U}}{\xi}\right)''_{x^2} - \sum_n \cdot \frac{\lambda_n}{n^2\pi^2} \xi^2 U''_{1x^2},$$

such that u = 0 for $y = \xi + \delta\xi$ where $\delta\xi$ will be given by*

$$\sum_n \left\{ (f_{1n} e^{c_n x} + f_{2n} e^{c_n x} + f_{0n}) [-2\varphi_{0n} c_n^2 (-1)^n \cdot \delta\xi] - \frac{\lambda_n}{c_n^2}\left(\frac{\overline{U}}{\xi}\right)''_{x^2} \xi - \frac{\lambda_n}{c_n^2} U''_{1x^2} \right\} = 0$$

[since $\sin c_n(\xi + \delta\xi) = \sin n\pi \cdot \cos c_n\delta\xi + \cos n\pi \cdot \sin c_n\delta\xi \cong (-1)^n \cdot c_n\delta\xi$].

A second boundary condition is the continuous connectivity between the field U interior to the boundary layer and the exterior field U_0, at the level $y = \xi + \delta\xi$ where u vanishes. Here, we obtain

$$\left(\frac{d\,U_{tot}}{dy}\right)_{\xi+\delta\xi} = \frac{\overline{U}}{\xi} + \sum_n (f_{1n} e^{c_n x} + f_{2n} e^{c_n x} + f_{0n}) [-2\varphi_{0n} c_n^2 (-1)^n + 2\gamma_n] = 0$$

since $\cos c_n(\xi + \delta\xi) = \cos n\pi \cdot \cos c_n\delta\xi - \sin n\pi \cdot \sin c_n\delta\xi = (-1)^n$. From this, we obtain the condition

$$\sum_n \left(f_n [-2\varphi_{0n} c_n^2 (-1)^n] - \frac{\lambda_n}{c_n^2}\left(\frac{\overline{U}}{\xi}\right)''_{x^2} \right) = -\frac{\overline{U}}{\xi}.$$

This can be resolved into other n such as

$$f_n [-2\varphi_{0n} c_n^2 (-1)^n] - \frac{\lambda_n}{c_n^2}\left(\frac{\overline{U}}{\xi}\right)''_{x^2} = -\varpi_n \cdot \frac{\overline{U}}{\xi} \quad \text{where} \quad \sum_n \varpi_n = 1.$$

From this, it follows that

$$(-1)^n \cdot 2\varphi_{0n} f_n = \frac{1}{c_n^2}\left\{ \varpi_n \frac{\overline{U}}{\xi} - \frac{\lambda_n}{c_n^2}\left(\frac{\overline{U}}{\xi}\right)''_{x^2} \right\} \cong \frac{1}{c_n^2} \varpi_n \cdot \frac{\overline{U}}{\xi}.$$

It remains to see what happens to $V_{tot}(0)$ and to $V_{tot}(\xi)$. /16

Thus,

$$V_{tot}(y) = -\left(\frac{\overline{U}}{\xi}\right)'_x \cdot \frac{y^2}{2} - U'_{1x} \cdot y - \sum_n c_n (f_{1n} e^{c_n x} - f_{2n} e^{-c_n x})$$

---

* We can thus find $\frac{\delta\xi}{\xi} \equiv - \frac{\left(\frac{\overline{U}}{\xi}\right)''_{x^2}}{\frac{\overline{U}}{\xi}} \sum_n \frac{\xi^2}{\pi^2 n^2}$ which consequently will be very small like $\left(\frac{\overline{U}}{\xi}\right)''_{x^2}$ but, within the scope of the approximations made, will be a finite quantity.

$$\times [2\,\varphi_{0_n} \cos c_n y + \alpha_n + \beta_n y + \gamma_n y^2]$$

$$= -\left(\frac{\overline{U}}{\xi}\right)'_x \cdot \frac{y^2}{2} - U'_{1_x} \cdot y - \sum_n \cdot \frac{c_n (f_{1_n} e^{c_n x} - f_{2_n} e^{-c_n x})}{f_{1_n} e^{c_n x} + f_{2_n} e^{-c_n x} + f_{0_n}}$$

$$\times \left[\frac{(-1)^n}{c_n^2}\left\{\varpi_n \frac{\overline{U}}{\xi} - \frac{\lambda_n}{c_n}\left(\frac{\overline{U}}{\xi}\right)''_{x^2}\right\} \cos c_n y\right]$$

$$+ \sum_n \frac{\lambda_n}{c_n^2}\left[\left\{\left(\frac{\overline{U}}{\xi}\right)'_x - \frac{1}{c_n^2}\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}\right\} - \frac{1}{2}\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} y^2 - U'''_{1_{x^3}} y\right]$$

(referring to the expressions for βf_2, $\gamma f'_x$ and $\alpha f'_x$ given above).

In $y = 0$, i.e., at the interface with the sublayer ϵ, we principally will have

$$V_{tot}(0) \cong - \sum_n \cdot \frac{c_n (f_{1_x} e^{c_n x} - f_{2_x} e^{-c_n x})}{f_{1_n} e^{c_n x} + f_{2_n} e^{-c_n x} + f_{0_n}} \left\{\frac{\varpi_n}{c_n^2} \cdot \frac{\overline{U}}{\xi} (-1)^n\right\}.$$

Similarly, at the upper border $y = \xi + \delta\xi$, we have

$$V_{tot}(\xi + \delta\xi) \cong -\left(\frac{\overline{U}}{\xi}\right)'_x \frac{\xi^2}{2} - U'_{1_x} \xi - \sum_n \cdot \frac{c_n (f_{1_n} e^{c_n x} - f_{2_n} e^{-c_n x})}{f_{1_n} e^{c_n x} + f_{2_n} e^{-c_n x} + f_{0_n}} \left\{\frac{\varpi_n}{c_n^2} \cdot \frac{\overline{U}}{\xi}\right\}.$$

It is definitely necessary that these quantities are very small (but nothing stipulates that they be zero).

It results from this that $f'_{n_x} \simeq 0$, i.e., that in the integration by parts with respect to $x = \Delta x$, it is necessary to put $f_{1_n} \cong f_{2_n}$. It also follows from this that $f_n(x)$ will develop very slowly with x (since the linear term in Δx vanishes in the main).

5. Form of the Solutions Relative to y

It is convenient to write

$$-2\,\varphi_{0n} f_n(x) = \frac{1}{c_n} \Phi_n(x) \qquad \text{and} \qquad \Phi_n = K_n \Phi_1.$$

Making use of the expression found previously for $2\varphi_{0_n} f_n$, we obtain

$$\Phi_n = -\left[\varpi_n \cdot \frac{\overline{U}}{\xi} - \frac{\lambda_n}{c_n}\left(\frac{\overline{U}}{\xi}\right)''_{x^2}\right] \frac{(-1)^n}{c_n^1} \cong -\frac{\overline{U}}{\xi} \cdot \frac{\varpi_n}{c_n} (-1)^n = -\frac{(-1)^n}{n\pi} \varpi_n (U_0 - U_1)$$

and

$$\Phi_1 = \frac{1}{c_1}\left[\frac{\overline{U}}{\xi}\varpi_1 - \frac{\lambda_1}{c_1}\left(\frac{\overline{U}}{\xi}\right)''_{x^2}\right] \cong \varpi_1 \frac{\overline{U}}{c_1 \xi} = \varpi_1 \frac{U_0 - U_1}{\pi}$$

since $c_n = \frac{n\pi}{\xi}$, $\overline{U} = U_0 - U_1$.

Finally,

$$-\sum_n (-1)^n n \Phi_n = -\Phi_1 \sum_n (-1)^n n K_n = \frac{\overline{U}}{\pi}$$

taking $\sum_n \overline{\omega}_n = 1$ into consideration.

Then, Φ_1 is written as

$$\Phi_1 = \frac{U_0 - U_1}{\pi A} \qquad \text{with} \qquad A = -\sum_n (-1)^n \cdot n K_n.$$

In addition, we will put $\sum_n n^3 K_n = B$, which are quantities to be used later in the text. These will be the constants of the problem*, yielding

$$U_{tot} = \frac{\overline{U}}{\xi} y + U_1 + \Phi_1 \sum_n n K_n \sin c_n y - y\left(\frac{\overline{U}}{\xi}\right)''_{x^2} \sum_n \cdot \frac{1}{c_n^2} - U''_{1x^2} \sum_n \cdot \frac{1}{c_n^2}.$$

5.1 Useful Forms of the Velocity Components

We wish to express that the "turbulent" state follows the laminar state to which it is connected in a continuous manner; let x_j be the connectivity segment.

Let the field U be the laminar Blasius field; it is necessary to write that this simultaneously represents the initial "turbulent" field. The n coefficients $\Phi_n(x_j) = \Phi_{n_j}$ of the Fourier expansion included in U_{tot} will result from this condition, since the Blasius field is known. Nevertheless, it is necessary to define the limit ϵ_j forming the interface with the sublayer. According to the definition of this sublayer, the gradient $U'y$ stops being constant for the coordinate Y. In the Blasius field, as indicated in Diagram I in Fig.16 (see Sect.21.6), this corresponds to the following:

$$U_{1j} \cong 0.66\, U_0, \qquad \frac{\epsilon_j}{\delta_j} \cong 0.365,$$

such that

* Since the number of terms n is finite, $\sum n^3 K_n$ will have some meaning (we will demonstrate that four terms are sufficient).

$$\frac{\xi_j}{\delta_j} \cong 0.635 = \sigma,$$

where δ_j is the thickness of the laminar boundary layer $\left(\text{taking } \delta_j \cong 5.5\sqrt{\frac{\nu x_j}{U_0}}\right)$.

Applying the preceding formulas, it is thus easy to calculate the quantities $\frac{1}{U_0}\,\Phi_n$ and K_n. /18

It will be found that these coefficients K_n are very small with respect to 1, while Φ_n are very small with respect to Φ_1. This indicates that the quantities $\overline{\omega}_n$ will be very small with respect to $\overline{\omega}_1$ which itself is very close to unity.

For $x > x_j$, the above-established relations show that $\Phi_n(x)$ can be derived from $\Phi_n(x_j)$ by

$$\Phi_n(x) \cong \Phi_n(x_j)\frac{U_0 - U_1(x)}{U_0 - U_{1j}},$$

where the quantities K_n remain constant. Thus, relative to y, the configuration of the fields U and V will be known as soon as the law $U_1(x)$ is known [and $\epsilon(x)$].

Consequently, the forms to be retained are as follows*

$$f'_{n_x} = c_n\left(f_{1_n} e^{c_n x} - f_{2_n} e^{-c_n x}\right), \qquad \sum_n \beta_n f_n = -U''_{1x^2}\sum_n \frac{1}{c_n^2},$$

$$\sum_n \gamma_n f_n(x) \cong -\left(\frac{\overline{U}}{\xi}\right)''_{x^2}\sum_n \cdot \frac{1}{c_n^2}, \qquad c_n = \frac{n\pi}{\xi};$$

$$u(x, y) = \sum_n \left[\Phi_n \sin c_n y + 2\gamma_n f_n \cdot y + \beta_n f_n\right] \cong \sum_n \Phi_n \cdot \sin c_n y;$$

$$\frac{\partial u}{\partial y} = \sum_n \left[c_n \Phi_n \cos c_n y + 2\gamma_n f_n\right] \cong \sum_n c_n \Phi_n \cdot \cos c_n y.$$

Finally,

$$v(x, y) = -\sum_n \left\{ c_n\left(f_{1_n} e^{c_n x} - f_{2_n} e^{-c_n x}\right)\left[2\varphi_{0_n} \cos c_n y + \alpha_n + \beta_n y + \gamma_n y^2\right]\right\}$$

---

* The terms in γ_n are connected with the concomitant presence, in the fundamental equation in ψ (see Sect.3.2), of the term in $\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}$ and, under the boundary conditions $y = \xi$, of the very small terms in $\Delta U(\xi)$ and $\frac{\partial \Delta U}{\partial y}$. These are negligible in the problem in y, but cannot be neglected in the problem in x.

$$= -\sum_n \left\} 2\,\varphi_{0_n}\, c_n\,(f_{1_n} e^{c_n x} - f_{2_n} e^{-c_n x}) \left[\cos c_n\, y + \frac{\alpha_n}{2\,\varphi_{0_n}}\right] - \frac{y^2}{2\,c_n^2}\cdot\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} - \frac{y}{c_n^2}\,U'''_{1_{x^3}} \right\{$$

or

$$v\,(x,y) = -\sum_n \left\} \frac{1}{c_n^2}\left[\left(\frac{\overline{U}}{\xi}\right)'_x - \frac{1}{c_n^2}\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}\right][\cos c_n\,y - 1] - \frac{y^2}{2\,c_n^2}\left(\frac{\overline{U}}{\xi}\right)'''_{x^3} - \frac{y}{c_n^2}\cdot U'''_{1_{x^3}} \right\{.$$

[See Sect.40.6, Diagram V in Fig.32, for the form of the fields U(Y) calculated in this manner.]

It should be mentioned here that it has been rationally possible to devise a form of the stationary field U(y), in the direct vicinity of a plane wall, quite analogous to that experimentally found for the turbulent boundary layer (at moderate velocities). This result was obtained without having to invoke the viscosity effects; it was sufficient to write the continuity condition and certain boundary conditions referring to the connectivity at the upper border (potential flow U_0) and lower border (sublayer ε where the viscous effects are localized). /19

6. Expressions of Rotation and Fundamental Relation at the Base of the Actual Boundary Layer

Let us next calculate

$$\Omega_{tot} = \frac{1}{2}\left[\frac{\partial V_{tot}}{\partial x} - \frac{\partial U_{tot}}{\partial y}\right] \cong -\frac{1}{2}\cdot\frac{\partial U_{tot}}{\partial y} = -\frac{1}{2}\left[\frac{\overline{U}}{\xi} + \frac{\partial u}{\partial y}\right]$$

Thus,

$$\Omega_{tot} \cong -\frac{1}{2}\left[\frac{\overline{U}}{\xi} + \sum_n \left\} c_n\,\Phi_n \cos c_n\,y + 2\,\gamma_n\,f_n \right\{\right].$$

Later in the text, we will have to write the fundamental relation $\frac{d\Omega_{tot}}{dt} = \nu \cdot \Delta^2\Omega_{tot}$ with the rotations at the base y = 0 of the actual boundary layer:

$$\frac{d\,\Omega_{tot}}{dt} = \frac{\partial\,\Omega_{tot}}{\partial x}\cdot U_{tot} + \frac{\partial\,\Omega_{tot}}{\partial y}\cdot V_{tot}, \qquad \Delta^2\,\Omega_{tot} = \frac{\partial^2\,\Omega_{tot}}{\partial x^2} + \frac{\partial^2\,\Omega_{tot}}{\partial y^2}$$

where

$$\frac{\partial\,\Omega_{tot}}{\partial x} = \frac{1}{2}\left[\frac{\partial^2 V_{tot}}{\partial x^2} - \frac{\partial^2 U_{tot}}{\partial y\,\partial x}\right] \cong -\frac{1}{2}\cdot\frac{\partial^2 U_{tot}}{\partial y\,\partial x}; \qquad \frac{\partial^2\,\Omega_{tot}}{\partial x^2} = \frac{1}{2}\left[\frac{\partial^3 V_{tot}}{\partial x^3} - \frac{\partial^3 U_{tot}}{\partial y\,\partial x^2}\right]$$

$$\frac{\partial \Omega_{tot}}{\partial y} = \frac{1}{2}\left[\frac{\partial^2 V_{tot}}{\partial x \partial y} - \frac{\partial^2 U_{tot}}{\partial y^2}\right], \qquad \frac{\partial^2 \Omega_{tot}}{\partial y^2} = \frac{1}{2}\left[\frac{\partial^3 V_{tot}}{\partial x \partial y^2} - \frac{\partial^3 U_{tot}}{\partial y^3}\right].$$

Since the continuity condition is satisfied

$$\frac{\partial V_{tot}}{\partial y} = -\frac{\partial U_{tot}}{\partial x}$$

or

$$\frac{\partial^2 V_{tot}}{\partial y^2} = -\frac{\partial^2 U}{\partial x \partial y}, \qquad \frac{\partial^3 V_{tot}}{\partial y^2 \partial x} = -\frac{\partial^3 U_{tot}}{\partial x^2 \partial y}$$

we thus have

$$\frac{\partial^2 U_{tot}}{\partial y \partial x} = \left(\frac{\bar{U}}{\xi}\right)'_x + \frac{\partial^2 u}{\partial y \partial x} = \left(\frac{\bar{U}}{\xi}\right)'_x + \psi'''_{y^2x} = \left(\frac{\bar{U}}{\xi}\right)'_x + \varphi''_{y^2} f'_x$$

$$= \left(\frac{\bar{U}}{\xi}\right)'_x - \sum_n [f'_{n_x}\{2\varphi_{0_n} c_n^2 \cos c_n y - 2\gamma_n\}].$$

At $y = 0$: /20

$$\left[\frac{\partial^2 U}{\partial x \partial y}\right]_0 = -\sum_n (f'_{n_x} c_n^2 2\varphi_{0_n} - 2 f'_{n_x}\gamma_n) = -\sum_n (-1)^n \varpi_n \left(\frac{\bar{U}}{\xi}\right)'_x - \sum_n \frac{\lambda_n}{c_n^2}\left(\frac{\bar{U}}{\xi}\right)'''_{x^3}.$$

$$\cong \left(\frac{\bar{U}}{\xi}\right)'_x - \frac{1}{c_n^2}\left(\frac{\bar{U}}{\xi}\right)'''_{x^3} {}^{*}.$$

$$\frac{\partial^2 U_{tot}}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} = \psi'''_{y^3} = \varphi'''_{y^3} \cdot f = \sum_n f_n 2\varphi_{0_n} c_n^3 \sin c_n y.$$

For $y = 0$: $\frac{\partial^2 u}{\partial y^2} = 0.$

$$\frac{\partial^3 U_{tot}}{\partial y \partial x^2} = \left(\frac{\bar{U}}{\xi}\right)''_{x^2} + \frac{\partial^3 u}{\partial y \partial x^2} = \left(\frac{\bar{U}}{\xi}\right)'''_{x^3} + \psi'''_{x^2y} = \left(\frac{\bar{U}}{\xi}\right)''_{x^2} + \varphi''_{y^2} f''_{x^2}.$$

For $y = 0$: $$\frac{\partial^3 U_{tot}}{\partial y \partial x^2} = \left(\frac{\bar{U}}{\xi}\right)'''_{x^3} - \sum_n (-1)^n \varpi_n \left(\frac{\bar{U}}{\xi}\right)''_{x^2} - \sum_n \frac{\lambda_n}{c_n^2}\left(\frac{\bar{U}}{\xi}\right)'''_{x^3}$$

$$\cong 2\left(\frac{\bar{U}}{\xi}\right)'''_{x^3} - \sum_n \frac{1}{c_n^2}\left(\frac{\bar{U}}{\xi}\right)'''_{x^3}.$$

$$\frac{\partial^3 U_{tot}}{\partial y^3} = \frac{\partial^3 u}{\partial y^3} = \psi''''_{y^4} = \varphi''''_{y^4} f = \sum_n f_n c_n^4 2\varphi_{0_n} \cos c_n y.$$

* If the complementary term in $2f'_{nx}\gamma_n$, representing the term of residual velocity (with respect to U_0) at the level $y = \xi$ is neglected, then $\frac{1}{c_n^2}\left(\frac{\bar{U}}{\xi}\right)'''_{x^3}$ will vanish. This approximation will be investigated later in the text.

$$\text{For } y=0: \quad \frac{\partial^3 U_{tot}}{\partial y^3} = -\sum_n c_n^3 \Phi_n.$$

$$\text{Finally, } U_{tot} = U_1, \; V_{tot} \cong 0.$$

6.1 General Equations of the Problem in x, in the Actual Boundary Layer

The fundamental equation at the base in question, multiplying each side of the equation by −1, will be written as

$$\frac{1}{2} U_1 \left(\frac{\partial^2 U_{tot}}{\partial y \, \partial x}\right)_0 = \frac{\nu}{2}\left[2 \cdot \left(\frac{\partial^3 U_{tot}}{\partial y \, \partial x^2}\right)_0 + \left(\frac{\partial^3 U_{tot}}{\partial y^3}\right)_0\right]$$

or

$$U_1 \left[\left(\frac{\overline{U}}{\xi}\right)'_x - \sum_n \frac{1}{2 c_n^2}\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}\right] \cong \nu \left[2\left(\frac{\overline{U}}{\xi}\right)''_{x^2} - \frac{1}{2}\sum_n c_n^3 \Phi_n\right]. \qquad \text{(I)}$$

[The terms in $\nu \dfrac{\partial^3 V_{tot}}{\partial x^3}$ are neglected as well as those in $-\nu \cdot \sum_n \dfrac{1}{c_n^2} \left(\dfrac{\overline{U}}{\xi}\right)''''_{x^4}$ and in $-\nu \sum_n \cdot \dfrac{1}{c_n^2} U''''_{1_{x^4}}$].

It should be noted that /21

$$\sum_n c_n^3 \Phi_n = \sum_n \frac{\pi^3}{\xi^3} n^3 \Phi_n = \frac{\pi^3}{\xi^3} B \, \Phi_1,$$

$$-\sum_n (-1)^n c_n \Phi_n = -\sum_n (-1)^n \frac{\pi}{\xi} n \, \Phi_n = \frac{\pi}{\xi} A \, \Phi_1,$$

and, consequently,

$$\sum_n c_n^3 \Phi_n = \pi^2 \cdot \frac{B}{A} \cdot \frac{\overline{U}}{\xi^3}.$$

It will be found that writing eq.(I) reduces to picking up the relation of definition of the stream function ψ (see Sect.3) at the base of the actual boundary layer, retaining the principal terms in ν. We will demonstrate below that actually, and within the scope of the adopted schematization, a discontinuity of the velocity gradient and of the rotation generally exists. In that case, the corresponding terms $\left(\text{even as a product with } \dfrac{\nu}{U_0 \xi}\right)$ may remain locally significant. Consequently, they must be retained in the analysis.

The next step is to expand eq.(I) for which the derivatives of $\dfrac{\overline{U}}{\xi}$ must be calculated, taking into consideration that the products of derivatives will be of a negligible infinitesimal order:

$$\left(\frac{\overline{U}}{\xi}\right)'_x=\frac{\overline{U}'_x}{\xi}-\frac{\overline{U}}{\xi}\cdot\frac{\xi'_x}{\xi},\qquad \left(\frac{\overline{U}}{\xi}\right)''_{x^2}=\frac{\overline{U}''_{x^2}}{\xi}-2\,\frac{\overline{U}'_x\,\xi'_x}{\xi^2}-\frac{\overline{U}}{\xi}\left(\frac{\xi''_{x^2}}{\xi}-2\,\frac{\xi'^2_x}{\xi^2}\right)\cong\frac{\overline{U}''_{x^2}}{\xi}-\frac{\overline{U}}{\xi}\cdot\frac{\xi''_{x^2}}{\xi}.$$

$$\left(\frac{\overline{U}}{\xi}\right)'''_{x^3}=\frac{\overline{U}'''_{x^3}}{\xi}-3\,\frac{\overline{U}''_{x^2}}{\xi}\cdot\frac{\xi'_x}{\xi}-3\,\frac{\overline{U}'_x}{\xi}\cdot\frac{\xi''_{x^2}}{\xi}+6\,\frac{\overline{U}'_x}{\xi}\cdot\frac{\xi'^2_x}{\xi^2}$$
$$+\frac{\overline{U}}{\xi}\left(\frac{6\,\xi'_x\,\xi''_{x^2}}{\xi^2}-4\,\frac{\xi'^3_x}{\xi^3}-\frac{\xi'''_{x^3}}{\xi}\right)\cong\frac{\overline{U}'''_{x^3}}{\xi}-\frac{\overline{U}}{\xi}\cdot\frac{\xi'''_{x^3}}{\xi}.$$

Finally, we have $\overline{U}'_x = -U'_{1x}$ since $\overline{U} = U_0 - U_1$. Consequently, the condition (I) will be successively written in the form

$$U_1\left[\cdot\frac{\overline{U}'_x}{\xi}-\frac{\overline{U}}{\xi^2}\cdot\xi'_x-\frac{\xi^2}{2\,\pi^2}\sum_n\frac{1}{n^2}\left(\frac{\overline{U}'''_{x^3}}{\xi}-\overline{U}\,\frac{\xi'''_{x^3}}{\xi^2}\right)\right]$$
$$=\nu\left[2\left\{\frac{\overline{U}''_{x^2}}{\xi}-\overline{U}\,\frac{\xi''_{x^2}}{\xi^2}\right\}-\frac{\pi^2}{2}\cdot\frac{B}{A}\cdot\frac{\overline{U}}{\xi^3}\right];$$

$$U_1\left[-\frac{U'_{1x}}{\xi}-\frac{U_0-U_1}{\xi^2}\,\xi'_x-\frac{\xi^2}{2\,\pi^2}\sum_n\cdot\frac{1}{n^2}\left(-\frac{U'''_{x^3}}{\xi}-(U_0-U_1)\,\frac{\xi'''_{x^3}}{\xi^2}\right)\right]$$
$$=\nu\left[2\left\{-\frac{U''_{1x^2}}{\xi}-(U_0-U_1)\,\frac{\xi''_{x^2}}{\xi^2}\right\}-\frac{\pi^2}{2}\cdot\frac{B}{A}\cdot\frac{U_0-U_1}{\xi^3}\right];$$

$$U_1\left[\xi'_x+\frac{U'_{1x}}{U_0-U_1}\,\xi-\frac{\xi^2}{2\,\pi^2}\sum_n\frac{1}{n^2}\left(\xi'''_{x^3}+\xi\,\frac{U'''_{1x^3}}{U_0-U_1}\right)\right]$$
$$=\nu\left[2\left\{U''_{1x^2}\cdot\frac{\xi}{U_0-U_1}+\xi''_{x^3}\right\}+\frac{\pi^2}{2}\cdot\frac{B}{A}\cdot\frac{1}{\xi}\right].$$

Let us multiply by $\frac{1}{U_1}$, whence /22

$$\xi'_x-\left\{\frac{2\,\nu}{U_1}\,\xi''_{x^2}\right\}-\frac{\xi^2}{2\,\pi^2}\sum_n\frac{1}{n^2}\,\xi'''_{x^3}$$
$$=\frac{\nu}{U_1}\left[\frac{\pi^2}{2}\cdot\frac{B}{A}\cdot\frac{1}{\xi}+\left\{2\,\xi\,\frac{U''_{1x^2}}{U_0-U_1}\right\}\right]-\frac{U'_{1x}}{U_0-U_1}\,\xi+\frac{\xi^2}{2\,\pi^2}\sum_n\frac{1}{n^2}\cdot\frac{\xi\,U'''_{1x^3}}{U_0-U_1}.$$

The terms containing products of the derivatives with respect to x and of $\frac{\nu}{U_1}$ (terms between $\{\}$) can be eliminated, finally yielding

$$\xi'_x-\frac{\sum_n\frac{1}{n^2}}{2\,\pi^2}\,\xi^2\,\xi'''_{x^3}=\frac{\nu}{U_1}\cdot\frac{\pi^2}{2}\cdot\frac{B}{A}\cdot\frac{1}{\xi}-\frac{\xi}{U_0-U_1}\left[U'_{1x}-\frac{\sum_n\frac{1}{n^2}}{2\,\pi^2}\,\xi^2\,U'''_{1x^3}\right] \qquad \text{(Ia)}$$

which will be the form to be used below.

6.2 Numerical Determinations

At the end of Part I*, we will give a numerical determination of the coefficients Φ_n, and of the quantities B, A, $\frac{B}{A}$ for a connectivity at the laminar Blasius stage in an arbitrary segment x_j.

Let us state from now on that, for $U_{1_j} = 0.65\ U_0$, the following applies:

$$\frac{1}{U_0}\Phi_{1_j} = 0.112, \quad \frac{1}{U_0}\Phi_{2_j} = 0.013^{2}, \quad \frac{1}{U_0}\Phi_{3_j} = +\,0.003^{8},$$

$$\frac{1}{U_0}\Phi_{4_j} = \frac{1}{U_0}\Phi_{5_j} \cong 0, \quad \frac{1}{U_0}\Phi_{6_j} = -\,0.000^{8};$$

$$A = 0.900, \quad B = 1.41^{5}, \quad \frac{B}{A} = 1.57, \quad \frac{1}{2\pi^2}\Sigma\frac{1}{n^2} = 0.075.$$

However, whereas the determination of A is accurate, that of B is less precise. Minor uncertainties relative to the coefficients $\frac{1}{U_0}\ \Phi_n$, for n of a higher order, have a considerable influence on this factor (let us recall that $B = \sum_n n^3 \frac{\Phi_n}{\Phi_1}$). Thus, $\frac{B}{A}$ is determined with a certain margin of indeterminacy.

For $U_1 \rightarrow X = 0.45\ U_0$, a value to which we must refer here, we obtain

$$\frac{1}{U_0}\Phi_1 \cong 0.176, \quad \frac{1}{U_0}\Phi_2 = 0.020^{7}, \quad \frac{1}{U_0}\Phi_3 = 0.006, \quad \frac{1}{U_0}\Phi_6 = -\,0.001,$$

with $\frac{B}{A}$ remaining unchanged.

* See Section 21.6 below.

CHAPTER II /23

BOUNDARY SUBLAYER

7. Structure

It is now necessary to determine the structural scheme on which the sublayer depends which, as should be recalled here, is the zone of very slight thickness ϵ comprised between the wall of the plate and the lower border of the actual boundary layer, where the velocity is U_1 (Fig.2).

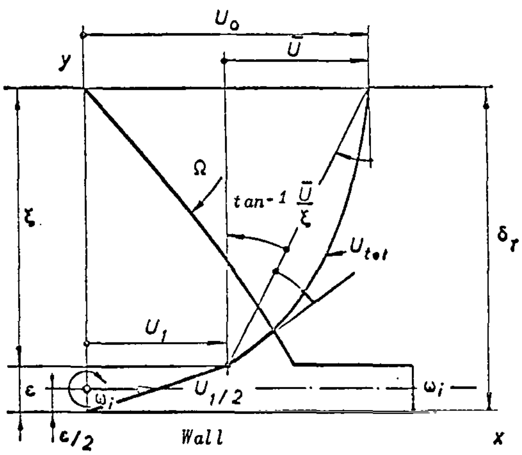

Fig.2

The boundary conditions, referring to this sublayer, will thus be as follows:

on the wall:

$$U_{tot} = 0, \qquad V_{tot} = 0;$$

at the border ϵ:

$$U_{tot} = U_1, \qquad V_{tot} \cong 0.$$

The simplest scheme to which we can refer here is that of a linear distribution (in y) of U_{tot}, from zero at the wall to U_1 at the border line. (A theoretical justification of this law will be given in Appendix IV.)

Of course, this will be only a tentative scheme, i.e., a procedure for introducing a fictitious sublayer equivalent to the real sublayer into the problem, provided that one can prove that this is not physically absurd, and explicitly /24 writing that - at least on the average - the fundamental relations are satisfied.

Since, for Y comprised between zero and ϵ*,

$$U_{tot} = \frac{Y}{\epsilon} U_1, \qquad V_{tot} \cong 0, \qquad \frac{\partial U_{tot}}{\partial Y} = \frac{U_1}{\epsilon} \text{ const.}$$

* The continuity condition, as before, yields

$$\frac{\partial V_{tot}}{\partial y} = -\left(\frac{U_1}{\epsilon}\right)'_x \cdot Y, \qquad V_{tot} = -\frac{Y^2}{2}\left(\frac{U_1}{\epsilon}\right)'_x$$

which remains very low for the double reason of ϵ being small and $\left(\frac{U_1}{\epsilon}\right)'_x$ being small.

The rotation is $\omega_1 = -\frac{1}{2} \cdot \frac{U_1}{\epsilon}$ which is also constant (relative to Y). Thus, a discontinuity of $\frac{\partial U_{tot}}{\partial Y}$ and of ω_1 on traversing the boundary U_1 will generally exist, except when ϵ is such that

$$\frac{U_1}{\varepsilon} = \sum_n [\Phi_n c_n \cos c_n y - 2\gamma_n / n].$$

This will be the particular case of the segment x_j of connectivity between laminar and turbulent flow, i.e., the case of the last segment where the flow ceases being of the Blasius type and is replaced by the "turbulent" type defined by the equations investigated in the previous Chapter*

From this, we derive the initial values of $U_1 = U_{1j}$, of $\epsilon = \epsilon_j$, and of $\xi = \xi_j$ as soon as – since x_j is assumed as given – the value δ_j and the field $U_{tot\,j}(x_j, Y)$ are known from the Blasius theory.

In the general case, the presence of a rotation discontinuity ω_1 at the boundary U_1 indicates the presence of particles rolling along the wall and undergoing intense rotations in its direct vicinity.

8. Discussion of the Validity of the Rotation Scheme

The first question raised in this respect is whether this rotation discontinuity along a boundary U_1 represents a reasonable physical image.

Let us assume a line of particles in rotation, moving at uniform velocity along a wall, with their viscous dissipation at each instant being compensated by the creation of new rotations for maintaining – at each point and at each instant – the derivative $\frac{\partial \omega(x, 0)}{\partial x}$ constant.

Thus, at a point I $(x = 0, y)$ fixed with respect to the wall and at a short distance from this wall, we will attempt to define the value of the rotation, diffused (by dissipation) by the series of vortical elements in contact with the wall.

8.1 Rotation Diffused at a Fixed Point in Space, by a Moving Vortex /25

Let us first consider a particle that entered into rotation in x_0 at the

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* In this particular case, the velocity field U which is a Blasius field satisfies, by elimination of the pressure between the Navier equations, the above-investigated equations; thus, the Blasius field is a particular case of a more general, although still approximate, solution of the Navier equations subject to approximations already less limitative than those of Blasius.

time zero, with an intensity $\omega_0(0)$ corresponding to a circulation γ_0.

This particle arrives at x at the time t, where $x = x_0 = U(t - t_0)$. At this instant, according to studies by Prandtl (Ref.1) on the dissipation of a vortical element, the particle induces a rotation (Fig.3) by diffusion in I(0, y):

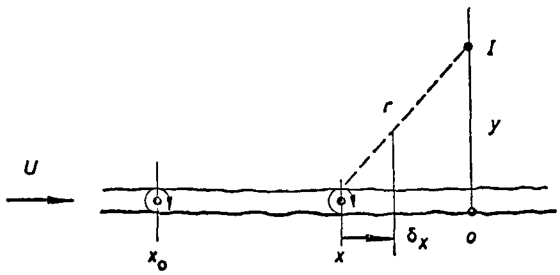

Fig.3

$$\omega_y(x, t) = \frac{A_0}{t - t_0} e^{-\frac{r^2}{4\nu(t-t_0)}}$$

where

$$r^2 = (-x)^2 + y^2.$$

Its intensity of rotation in x, at the instant t, is as follows:

$$\omega_0(t) = \frac{A_0}{t - t_0} = \frac{A_0 U}{x - x_0},$$

where the subscript 0 of ω indicates that rotations of the particles of the line itself are involved, i.e., rotations such as r = 0.

It should be recalled here that A_0 is linked to the circulation γ_0 by the following statement, resulting from the definition of the latter:

$$\gamma(t) = 4\pi\int_0^r \omega(r)\, r\, dr = 8\pi\nu A_0\left(1 - e^{-\frac{r^2}{4\nu(t-t_0)}}\right).$$

As $r \to \infty$ or $t \to t_0$, it also follows that $\gamma(t) \to 8\pi\nu A_0$ such that $\gamma_0 = 8\pi\nu A_0$.

As soon as $\omega_0(t)$, starting from x, executes the step δx such as $\delta x = U\delta t$, this intensity, by dissipation, undergoes the following variation of rotation:

$$\delta\omega_0(x) = \frac{-A_0 U}{(x - x_0)^2}\delta x.$$

So as to have the rotation ω_0 retain its initial value, it is necessary that a compensatory rotation of the loss $\delta\omega_0$ is generated on the same particle during its transit δx, namely /26

$$\delta^2\omega_0(x) = \frac{\delta A_0 . U}{\delta x},$$

such that

$$-A_0\frac{U}{(x-x_0)^2}\delta x+\delta A_0\frac{U}{\delta x}=0,$$

whence

$$\delta A_0=\frac{A_0}{(x-x_0)^2}\delta x^2.$$

Here, δA_0 appears as infinitely small (of the second order in δx) with respect to $A_0\delta x$ of the first order.

In I(0, y), located outside the line, the variation in rotation induced by diffusion of ω, during its transit δx, will then have been

$$\begin{aligned}\delta\omega_y(x)+\delta^2\omega_y(x)&=\frac{A_0 U}{x-x_0}\left[\left\{-\frac{\delta x}{x-x_0}-\frac{\partial}{\partial x}\left(\frac{(-x)^2+y^2}{x-x_0}\cdot\frac{U}{4\nu}\right)\delta x\right\}e^{-\frac{U}{4\nu}\cdot\frac{x^2+y^2}{x-x_0}}\right.\\&\qquad\left.+\frac{\delta x}{x-x_0}e^{-\frac{U}{4\nu}\cdot\frac{x^2+y^2}{\delta x}}\right]\\&=-\frac{A_0 U}{(x-x_0)^2}\delta x\left[\left\{1+\frac{U}{4\nu}\cdot\frac{2x(x-x_0)-(x^2+y^2)}{x-x_0}\right\}e^{-\frac{U}{4\nu}\cdot\frac{x^2+y^2}{x-x_0}}\right],\end{aligned}$$

since the term in $e^{-\frac{U}{4\nu}\cdot\frac{x^2+x^2}{\delta y}}$ can be neglected $\left(\frac{x^2+y^2}{\delta x}\right.$ being infinitely large$\left.\right)$.

The rotation induced in I by the element ω_0, from its origin in x_0 until the instant of its reaching x, will have the following expression:

$$\begin{aligned}\omega_y(x)&=-A_0 U\int_{x=x_0}^{x}\left[\frac{U}{4\nu}\cdot\frac{2x(x-x_0)+(x^2+y^2)}{(x-x_0)^2}+1\right]e^{-\frac{U}{4\nu}\cdot\frac{x^2+y^2}{x-x_0}}\cdot dx\\&=\frac{A_0 U}{x-x_0}e^{-\frac{U}{4\nu}\cdot\frac{x^2+y^2}{x-x_0}}.\end{aligned}$$

8.2 Rotations Diffused by a Vortex Line in Dissipation

Let us now combine a permanent line of elements similar to those considered above and all originating in x_0, with the density of rotation being uniform so that also the density of the quantities A_0 generating these elements at the origin x_0 will be uniform (for this, we will denote it by $A'_{0_{x_\nu}}$):

$$\frac{\partial A_0}{\partial t}=U A'_{0x_\nu}=\text{const}.$$

The intensity of rotation induced in I is expressed as follows: /27

$$\Sigma\, \omega_y (0, x) = A'_{0x_0}\, U \int_{x_i = x_0}^{x} e^{-\frac{U}{4\nu}\cdot\frac{x^2+y^2}{x_i - x_0}} \cdot \frac{dx_i}{x_i - x_0} = A'_{0\varkappa_0}\, U \int_{\varkappa_i = 0}^{\varkappa} e^{-\frac{Uy}{4\nu}\cdot\frac{1+\varkappa_i^2}{\varkappa_i - \varkappa_0}} \cdot \frac{d\varkappa_i}{\varkappa_i - \varkappa_0},$$

when putting $\varkappa_1 = \dfrac{x_1}{y}$.

It is necessary to assume $\varkappa_0$ and $\varkappa$ to be very large, i.e., $\varkappa_0 \to -\infty$, $\varkappa \to +\infty$ since y is assumed as very small, such that we can put $\varkappa_0 = -|\varkappa_0|$, $\varkappa = +|\varkappa_0|$ and cause $|\varkappa_0|$ to tend to infinity. Let us also put $\varkappa = \mathfrak{R} \cdot z$ which is a variable complex*: The integral in z is

$$\int_{AA'} e^{-\frac{Uy}{4\nu}\cdot\frac{1+z^2}{z+|\varkappa_0|}} \cdot \frac{dz}{z + |\varkappa_0|}.$$

It is obvious that $-|\varkappa_0|$ is the only (essential) singular point of the function under the integral sign.

Thus, this function is regular along the contour formed by AA′ and the half-circle erected on AA′, except at A which is surrounded by a small quarter-circle of radius δr (Fig.4).

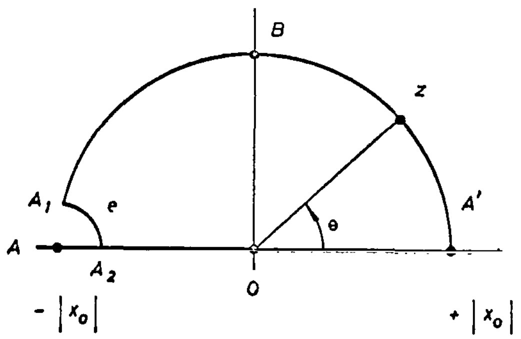

Fig.4

Let us form

$$\frac{z}{z + |\varkappa_0|}\, e^{-\frac{Uy}{4\nu}\cdot\frac{1+z^2}{z+|\varkappa_0|}}.$$

As $|z| \to \infty$ like $|\varkappa_0|$, the half-circle ABA′ increases indefinitely, and this quantity tends to zero since the exponent tends to zero like $e^{-\frac{U}{4V}\, y, z}$.

Thus, except for the small quarter circle in A of radius δr, the integral - which is zero over the entire contour and zero over the half-circle - is zero on A_1A_2A'.

Let us further investigate the small-quarter circle in A. We can write the integral as follows:

$$\int_c e^{-\frac{Uy}{4\nu}\cdot\frac{1}{z+|\varkappa_0|}} \cdot e^{-\frac{Uy}{4\nu}\cdot\frac{z^2}{z+|\varkappa_0|}} \cdot \frac{dz}{z + |\varkappa_0|}$$

* In what follows, $\mathfrak{R}$ is used for Re (Reynolds number), as in original.

and put $z = -|\varkappa_0| + \zeta$ from which it follows that $dz = d\zeta$; finally, /28

$$z + |x_0| = \zeta = \delta r \, . \, e^{\overset{\circ}{i}\varphi}.$$

This will yield

$$\int_c e^{-\frac{Uy}{4\nu}\cdot\frac{1}{\zeta}} \cdot e^{-\frac{Uy}{4\nu}\cdot\frac{(\zeta-|x_0|)^2}{\zeta}} \cdot \frac{d\zeta}{\zeta}.$$

As $|\varkappa_0| \to \infty$, it is always possible to select $|\varkappa_0|$ sufficiently large so that, no matter what δr might be, the modulus of

$$\left[e^{-\frac{Uy}{4\nu}\cdot\frac{1}{\zeta}} \cdot e^{-\frac{U'y}{4\nu}\cdot\frac{(\zeta-|x_0|)^2}{\zeta}}\right]$$

will be smaller than an arbitrary value ϵ. Then,

$$\int_c < \varepsilon \int_c \frac{d\zeta}{\zeta}.$$

Since

$$\int_{\zeta=0}^{\overset{\circ}{i}\frac{\pi}{2}} \frac{d\zeta}{\zeta} = \left| \operatorname{Log} \zeta \right|_0^{\overset{\circ}{i}\frac{\pi}{2}} = \overset{\circ}{i}\frac{\pi}{2},$$

$P \int_c < \epsilon i \frac{\pi}{2}$ tends to zero as ϵ tends to zero. The integral on the quarter circle thus will also be zero as soon as $|\varkappa_0| \to \infty$.

8.3 Application to the Sublayer Case

Consequently, provided that $\frac{y}{|x_0|}$ is sufficiently small, the rotation diffused in y can be neglected for the rotation of constant density existing in the vortex line in whose neighborhood the point I is located.

Thus, a rapid evolution of the density of rotation is possible in the neighborhood of a vortex sheet, and the discontinuity of $\omega(y)$ which our particular scheme incorporates on crossing the border line U_1 is by no means abnormal.

However, it is also obvious that, if the layer of uniform rotation density does not extend far upstream or far downstream of the point I under consideration, the vortical diffusion will become noticeable.

This would explain that, at increasing vortical density, i.e., whenever new rotations become superposed to the layer of uniform density, everything happens as though this layer would incorporate new particles which it would cause to rotate.

In the highly approximate concept used here, in which all variations of the quantities with x are assumed as very slow, this phenomenon can be neglected. /29

9. Constancy of the Gradient U'_y

Another question is raised in this respect, namely that of defining whether the Navier-Stokes equations - for a row of particles in rotation - are compatible with a distribution U such as $\frac{\partial U}{\partial y}$ = const in the thickness of the particle line.

Let us resume the reasoning of the preceding pages, applying them to a calculation of the induced velocities.

A rotation generated at the time t_0 will induce, at the time t and at a distance r from the nucleus, a tangential induced velocity

$$\mathfrak{U}(t, r) = \frac{8\pi\nu A_0}{2\pi r}\left[1 - e^{-\frac{1}{4\nu}\cdot\frac{r^2}{t-t_0}}\right]$$

(see Prandtl, loc. cit.) where $(t - t_0)U = x - x_0$.

For a path $\delta x = U \cdot \delta t$, the following induced velocity variation appears:

$$\delta\mathfrak{U} = -\frac{8\pi\nu A_0}{2\pi r}\cdot\left[\frac{r^2 U}{4\nu(x-x_0)^2}\, e^{-\frac{r^2 U}{4\nu(x-x_0)}}\right]\delta x.$$

Simultaneously, we have δA_0 to compensate the loss of diffused rotation such that

$$\delta A_0 = \frac{A_0}{(x-x_0)^2}\,\delta x^2,$$

whence

$$\delta^2\mathfrak{U} = \frac{8\pi\nu\,\delta A_0}{2\pi\cdot r}\left[1 - e^{-\frac{r^2 U}{4\nu\delta x}}\right] = \frac{A_0}{(x-x_0)^2}\cdot\frac{8\pi\nu}{r\,2\pi}\cdot\delta x^2$$

$\left(e^{-\frac{r^2 U}{4\nu\delta x}}\text{ being zero}\right)$. Here, $\delta^2\mathfrak{U}$ is of the second order infinitesimal in δx. This leaves only $\delta\mathfrak{U}$.

Due to the single vortex ω_0, it follows that

$$\mathfrak{U} = -\frac{8\pi\nu A_0}{2\pi\cdot r}\int_{x=x_0}^{x}\frac{r^2 U}{4\nu(x-x_0)^2}\, e^{-r^2\frac{U}{4\nu(x-x_0)}}\,dx = \frac{8\pi\nu A_0}{2\pi r}\left(1 - e^{-\frac{r^2 U}{4\nu(x-x_0)}}\right).$$

The totality of rotations ω_0, all originating in x_0 at uniform density, will generate, with $r = \sqrt{x_i^2 + y^2}$,

$$\mathfrak{U}_{tot} = \Sigma \mathfrak{U} = \frac{8\pi A'_{0x_0}}{2\pi}\int_{x_l=x_0}^{x} \frac{1-e^{-\frac{U}{4\nu}\cdot\frac{y^2+x_l^2}{x_l-x_0}}}{\sqrt{y^2+x_l^2}}\cdot dx_l.$$

According to the coordinates x and y, the velocity components u and v are /30
obtained by projection of $\mathfrak{U}$ (Fig.5).

At $x = 0$ and since $u = \mathfrak{U} \cdot \frac{y}{\sqrt{x_i^2 + y^2}}$, $v = \mathfrak{U} \cdot \frac{x_i}{\sqrt{x_i^2 + y^2}}$, we will have

$$u_{tot} = \Sigma\, u = 4\nu A'_{0x_0}\int_{x_l=x_0}^{x} \frac{1-e^{-\frac{U}{4\nu}\cdot\frac{y^2+x_l^2}{x_l-x_0}}}{\sqrt{x_l^2+y^2}}\cdot\frac{y}{\sqrt{x_l^2+y^2}}\, dx_l,$$

$$v_{tot} = \Sigma\, v = 4\nu A'_{0x_0}\int_{x_l=x_0}^{x} \frac{1-e^{-\frac{U}{4\nu}\cdot\frac{y^2+x_l^2}{x_l-x_0}}}{\sqrt{x_l^2+y^2}}\cdot\frac{x_l}{\sqrt{x_l^2+y^2}}\, dx_l.$$

Let us put $\varkappa_i = \frac{x_i}{y}$ so that an integration interval is defined such that $x = -x_0$ and $\frac{y}{x_0}$ are very small, i.e., that $|x_0|$ is large (for example, ± 100). Graphical integration will then be easy, yielding

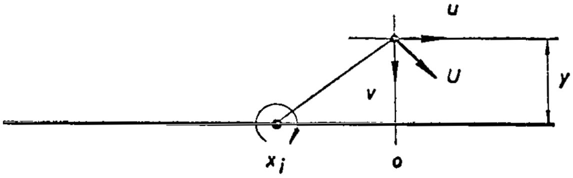

Fig.5

$$u_{tot} = 4\nu A'_{0\varkappa_0}\int_{-\varkappa_0}^{\varkappa_0} \frac{1-e^{-\frac{U y}{4\nu}\cdot\frac{1+\varkappa_l^2}{\varkappa_l-\varkappa_0}}}{\varkappa_l^2+1}\, d\varkappa_l,$$

$$v_{tot} = 4\nu A'_{0\varkappa_0}\int_{-\varkappa_0}^{\varkappa_0} \frac{1-e^{-\frac{U y}{4\nu}\cdot\frac{1+\varkappa_l^2}{\varkappa_l-\varkappa_0}}}{\varkappa_l^2+1}\, \varkappa_l\, d\varkappa_l.$$

The accompanying table and diagram show, at different values of $\frac{U\cdot y}{4\nu} = X$, the evolution of $\frac{U_{tot}}{4\nu A'_{x_0}} = Z$ (Fig.6).

This diagram indicates that $\frac{Z}{X}$ is constant in the domain $0 < X \leq 5$. This quantity, to within a constant factor $\frac{1}{A'_{0x_0}}$, measures the gradient $\frac{\partial U_{tot}}{\partial y}$.

(The superposition of several lines of elements in rotation with $\frac{\partial A_0}{\partial y} \cong 0$, can only thicken the domain with constant gradient $\frac{\partial U_{tot}}{\partial y}$.)

It should be noted here that $(U_{tot})'_r \cong (U_{tot})'_y$ = const represents the fact that the material elements located on the radius vector from 0 to y are subject to the same rotation and belong to a solid entity in rolling motion; this defines the dimension of what one might call the "particle". According to the preceding result, where $(U_{tot})'_r \cong (U_{tot})'_y$ = const for $0 < X \leq 5$, everything proceeds as though the dimension of the particle were $\bar{y} \leq 5 \cdot \frac{4\nu}{U} = \frac{\nu}{U} 20$. /31

$X = Uy/4\nu$	0	1	2	5	10	20
$Z = U_{Tot}/4\nu A'_{0x_0}$	0	0.09	0.19^5	0.48^5	0.80^5	1.13

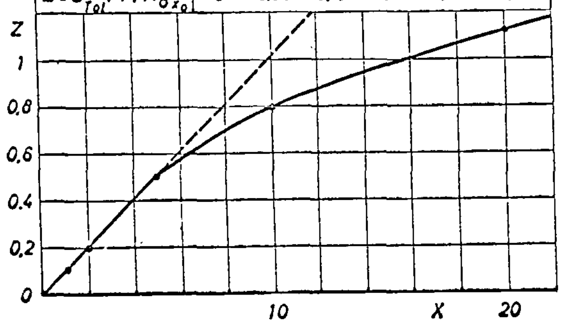

Fig.6

For U = 29 m/sec, we have

$\bar{y} \leq 10 \cdot 10^{-6}$ MKS (10 μ), in air.

For U = 87 m/sec, we have

$\bar{y} \leq 3.3 \cdot 10^{-6}$ MKS (3.3 μ), in air.

Although they are rather small, these dimensions are large with respect to the radius of a molecular volume (of the order of 10^{-9} MKS) and are large even with respect to the mean free path (of the order of 10^{-7}).

10. Fundamental Relation in the Sublayer

Since the scheme adopted for the sublayer cannot be contradicted from the viewpoint of physical possibilities, since here the sublayer is considered in its entity, the drive velocity of the particles is the mean velocity

$$\frac{dx}{dt} = \frac{U_1}{2} \quad \text{and} \quad \frac{dY}{dt} \cong 0,$$

no matter what x might be. Thus, $\frac{\partial V}{\partial x} = 0$.

The rotation is $\omega_i = -\frac{1}{2} \cdot \frac{\partial U}{\partial Y} = -\frac{1}{2} \cdot \frac{U_1}{\epsilon}$, where $\frac{\partial \omega_i}{\partial y} = 0$. /32

Applying the fundamental relation to the rotations, we obtain (see Sect.2.1)

$$\frac{U_1}{2} \cdot \frac{\partial \omega_i}{\partial x} = \nu \cdot \frac{\partial^2 \omega_i}{\partial x^2},$$

i.e.,

$$\frac{U_1}{2}\left(\frac{U_1}{\epsilon}\right)'_x = \nu \cdot \left(\frac{U_1}{\epsilon}\right)''_{x^2} \qquad \text{(II)}$$

which is the sought relation.

It should be noted that this relation is approximate, because of the very nature of the scheme used here, so that its reliability is not absolute.

CHAPTER III /33

THE PROBLEM RELATIVE TO THE TANGENTIAL COORDINATE x

11. Third Relation for Moment of Momentum Losses

In the preceding Chapters, we demonstrated that the problem, in each segment x, was determined with respect to y as soon as $U_1(x)$, $\xi(x)$, $\epsilon(x)$ were given.

Now it is necessary to calculate these three functions of x, which means that we require three distinct relations.

Two of these are already known; they are the fundamental relations of formation of rotations [eqs.(I) and (II)], of which one [eq.(I)] is written at the base of the actual boundary layer and the other [eq.(II)] at the interior of the sublayer, for characterizing the totality of its component.

Writing these equations simultaneously is necessary to express that, despite the discontinuities of the velocity gradient U'_y and of the rotation which is included in our scheme, compatibility of the two flows and compatibility of these flows with the general complete (Navier) equations will exist on traversing the border line U_1 between the layers*.

Because of this fact, the relations in question are quite distinct and constitute two functional conditions.

The third relation is classically derived from considerations of impulse in space, by evaluating the momentum losses existing in each segment x; on deriving the obtained expressions with respect to x, we will - by definition - obtain a first evaluation of the shearing force τ_0 along the wall; it will be sufficient to identify this with the general expression of local shear $\tau_0 = \mu\left[\frac{\partial U_{tot}}{\partial y}\right]_0$, derived from a consideration of the configuration of the velocity field, for obtaining the third sought condition. Here, $\left[\frac{\partial U_{tot}}{\partial y}\right]_0 = \frac{U_1}{\epsilon}$ and $\mu = \rho\nu$.

11.1 Expressions for the Loss of Moment of Momentum /34

Using the tangential velocity spectra determined previously, let us calculate the losses of moment of momentum

$$q = q_\epsilon + q_\xi = \rho\int_0^{\epsilon+\xi} U_{tot}\,(U_0 - U_{tot})\,dY$$

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* Because of the discontinuity of U'_y at the border line, the terms in ψ''''_{y4} become locally significant.

in each segment x.

In the sublayer, we have $U_{tot} = \frac{U_1}{\epsilon} Y$ so that

$$q_\epsilon = \rho \int_0^\epsilon \frac{U_1}{\epsilon} Y \left(U_0 - \frac{U_1}{\epsilon} \right) dY = \rho\, U_0^2 \cdot \frac{U_1}{U_0} \cdot \frac{\epsilon}{2} \left[1 - \frac{2}{3} \cdot \frac{U_1}{U_0} \right].$$

In the actual boundary layer,

$$U_{tot} = U_1 + \frac{\overline{U}}{\xi} y + \sum_n [\Phi_n \sin c_n y + 2\gamma_n f_n y] \cong U_1 + \frac{\overline{U}}{\xi} y + \sum_n \Phi_n \sin c_n y$$

whence

$$q_\xi = \rho \int_0^\xi \left\{ \frac{\overline{U}}{\xi} y + U_1 + \sum_n \Phi_n \sin n \frac{\pi}{\xi} y \right\} \left[U_0 - \left\{ \frac{\overline{U}}{\xi} y + U_1 + \sum_n \Phi_n \sin n \frac{\pi}{\xi} y \right\} \right] dy$$

$$= \rho \left[U_0 \left(\frac{\overline{U}}{\xi} \cdot \frac{y^2}{2} + U_1 y \right)_0^\xi - \left\{ \left(\frac{\overline{U}}{\xi} \right)^2 \cdot \frac{y^3}{3} + U_1^2 y + 2 \frac{\overline{U}}{\xi} U_1 \cdot \frac{y^2}{2} \right\}_0^\xi \right.$$

$$+ \sum_n \Phi_n \int_0^\xi (U_0 - 2U_1) \sin n \frac{\pi}{\xi} y \cdot \frac{d\left(n \frac{\pi}{\xi} y\right)}{n \frac{\pi}{\xi}} - 2 \sum_n \Phi_n \int_0^\xi \frac{\overline{U}}{\xi} \cdot \frac{\pi n}{\xi} y \sin \frac{n\pi}{\xi} y \frac{d\left(n \frac{\pi}{\xi} y\right)}{\left(\frac{n\pi}{\xi}\right)}$$

$$\left. - \sum_n \sum_p \Phi_n \Phi_p \int_0^\xi \sin n \frac{\pi}{\xi} y \cdot \sin p \frac{\pi}{\xi} y \cdot \frac{d\left(n \frac{\pi}{\xi} y\right)}{n \frac{\pi}{\xi}} - \sum_n \Phi_n^2 \int_0^\xi \sin^2 \frac{n\pi}{\xi} y \cdot \frac{d\left(n \frac{\pi}{\xi} y\right)}{n \frac{\pi}{\xi}} \right].$$

Let us put $\theta = \frac{\pi}{\xi} y$.

When y varies from 0 to ξ, also θ will vary from 0 to π, such that

$$\int_0^\xi \sin n \frac{\pi}{\xi} y \, d\left(n \frac{\pi}{\xi} y\right) = n \int_0^\pi \sin n\theta \cdot d\theta = -n \left| \frac{\cos n\theta}{n} \right|_0^\pi = \begin{cases} 2 \text{ if } n \text{ is odd,} \\ 0 \text{ if } n \text{ is even;} \end{cases}$$

$$\int_0^\xi \frac{n\pi}{\xi} y \sin n \frac{\pi}{\xi} y \cdot d\left(n \frac{\pi}{\xi} y\right) = n^2 \int_0^\pi \theta \sin n\theta \cdot d\theta = -n^2 \left| \theta \frac{\cos n\theta}{n} \right|_0^\pi + n \int_0^\pi \cos n\theta \, d\theta;$$

$$= -n^2 \pi \cdot (-1)^n.$$

$$\int_0^\xi \sin n \frac{\pi}{\xi} y \cdot \sin p \frac{\pi}{\xi} y \cdot d\left(n \frac{\pi}{\xi} y\right) = n \int_0^\pi \sin n\theta \cdot \sin p\theta \cdot d\theta$$

$$= -\frac{n}{2} \int_0^\pi [\cos(n+p)\theta - \cos(n-p)\theta] \, d\theta = 0;$$

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$$\int_0^\xi \sin^2 \left(n \frac{\pi}{\xi} y\right) \cdot d\left(n \frac{\pi}{\xi} y\right) = \frac{n}{2} \int_0^\pi (1 - \cos 2n\theta) \, d\theta = n \frac{\pi}{2}.$$

Then, the loss of moment of momentum q_ξ becomes

$$q_\xi \cong \rho\left[U_0\left(\frac{\overline{U}}{\xi}\cdot\frac{\xi^2}{2}+U_1\xi\right)-\left\{\left(\frac{\overline{U}}{\xi}\right)^2\cdot\frac{\xi^3}{3}+U_1^2\xi^2+2\frac{\overline{U}}{\xi}U_1\frac{\xi^2}{2}\right\}\right.$$
$$\left.+2(U_0-2U_1)\sum_n\frac{\Phi}{2p+1}\cdot\frac{1}{n\frac{\pi}{\xi}}-\frac{\overline{U}}{\xi}\cdot\frac{2}{\left(n\frac{\pi}{\xi}\right)^2}\sum_n\Phi_n\left(-(-1)^n\cdot n\pi\right)-\sum_n\Phi_n^2\cdot\frac{n\pi}{2}\cdot\frac{1}{\frac{n\pi}{\xi}}\right],$$

i.e.,

$$q_\xi=\rho U_0^2\cdot\xi\left[\frac{1}{2}\left(1+\frac{U_1}{U_0}\right)-\left(\frac{1}{3}\left(1-\frac{U_1}{U_0}\right)^2+\frac{U_1}{U_0}\right)+\frac{2}{\pi}\left(1-2\frac{U_1}{U_0}\right)\cdot\frac{1}{U_0}\sum_n\frac{\Phi_{2n+1}}{2n+1}\right.$$
$$\left.+(-1)^n\cdot\frac{2}{\pi}\left(1-\frac{U_1}{U_0}\right)\frac{1}{U_0}\sum_n\frac{\Phi_n}{n}-\frac{1}{2U_0^2}\sum_n\Phi_n^2\right].$$

Since $(q_\varepsilon + q_\xi)$ must be derived with respect to x, the following will appear:

$$\frac{\partial q_\varepsilon}{\partial x}=\frac{\partial q_\varepsilon}{\partial\varepsilon}\cdot\varepsilon'_x+\frac{\partial q_\varepsilon}{\partial U_1}\cdot U'_{1x}=\rho U_0^2\left[\frac{1}{2}\cdot\frac{U_1}{U_0}\left(1-\frac{2}{3}\frac{U_1}{U_0}\right)\cdot\varepsilon_x'+\frac{U'_{1x}}{U_0}\cdot\frac{\varepsilon}{2}\left(1-\frac{4}{3}\frac{U_1}{U_0}\right)\right]$$
$$\cong\rho U_0^2\frac{1}{2}\cdot\frac{U_1}{U_0}\left(1-\frac{2}{3}\cdot\frac{U_1}{U_0}\right)\varepsilon'_x \qquad \text{(since } \epsilon \text{ is small).}$$

Similarly,

$$\frac{\partial q_\xi}{\partial x}=\frac{\partial q_\xi}{\partial\xi}\cdot\xi'_x+\frac{\partial q_\xi}{\partial U_1}\cdot U'_{1x}.$$

We will put

$$\frac{\partial q_\xi}{\partial\xi}=\rho U_0^2\mathcal{A}$$

with

$$\mathcal{A}=\left[\frac{1}{2}\left(1-\frac{U_1}{U_0}\right)-\frac{1}{3}\left(1-\frac{U_1}{U_0}\right)^2+\frac{2}{\pi}\left(1-2\frac{U_1}{U_0}\right)\sum_n\cdot\frac{\frac{1}{U_0}\Phi_{2n+1}}{2n+1}\right.$$
$$\left.+\frac{2}{\pi}\left(1-\frac{U_1}{U_0}\right)\sum_n(-1)^n\cdot\frac{\frac{1}{U_0}\Phi_n}{n}-\sum_n 2U_0^2\Phi_n^2\right],$$

$$\frac{\partial q_\xi}{\partial U_1}=\rho U_0^2\cdot\xi\cdot\frac{\mathcal{B}}{U_0}$$

and with /36

$$\mathcal{B} = -\frac{1}{2}+\frac{2}{3}\left(1-\frac{U_1}{U_0}\right)-\frac{4}{\pi}\sum_n \frac{\frac{1}{U_0}\Phi_{2n+1}}{2n+1}-\frac{2}{\pi}\sum_n (-1)^n\cdot\frac{\frac{1}{U_0}\Phi_n}{n}.\ *$$

Thus, $\mathcal{A}$ and $\mathcal{B}$ appear as functions of $\frac{U_1}{U_0}$ and $\frac{U_1}{U_0}$ alone since, as should be recalled here, the quantities Φ_n are written in the form

$$\Phi_n = \Phi_1 \cdot K_n,$$

where K_n are constants while $\Phi_1 \cong \frac{U_0 - U_1}{A\pi}$ $\left(\text{i.e., } \frac{1}{U_0}\ \Phi_1 \cong \frac{1-\frac{U_1}{U_0}}{A\pi}\right)$.

11.2 Application to the Third Relation

The relation for derivatives of the losses of moment of momentum will thus be

$$\tau = \rho\nu\frac{U_1}{\varepsilon} = \frac{\partial}{\partial x}(q_\varepsilon + q_\xi)$$

$$= \rho U_0^2\cdot\left[\mathcal{A}\,\xi'_x + \left(\mathcal{B}\,\xi + \frac{\varepsilon}{2}\left(1-\frac{4}{3}\cdot\frac{U_1}{U_0}\right)\right)\frac{U'_{1x}}{U_0} + \frac{U_1}{U_0}\cdot\frac{\varepsilon'_x}{2}\left(1-\frac{2}{3}\cdot\frac{U_1}{U_0}\right)\right].$$

Since, by definition, the coefficient of local friction C_f^* is expressed as

$$C_f^* = \frac{\tau}{\frac{\rho}{2}U_0^2} = 2\,\frac{\nu}{U_0}\cdot\frac{U_1}{U_0}\cdot\frac{1}{\varepsilon},$$

it follows that

$$\varepsilon = \frac{2\nu}{U_0}\cdot\frac{U_1}{U_0}\cdot\frac{1}{C_f^*} \quad \text{and} \quad \varepsilon'_x = +\frac{2\nu}{U_0}\left[\frac{U'_{1x}}{U_0}\cdot\frac{1}{C_f^*} - \frac{U_1}{U_0}\cdot\frac{C^{*\prime}_{fx}}{C_f^{*2}}\right]. \qquad \text{(IV)}$$

Finally, the equation of derivatives of the losses of moment of momentum will be written in the form

$$C_f^* = 2\left[\mathcal{A}\,\xi'_x + \frac{1}{2}\cdot\frac{U_1}{U_0}\left(1-\frac{2}{3}\cdot\frac{U_1}{U_0}\right)\left\{\frac{U'_{1x}}{U_0}\cdot\frac{1}{C_f^*} - \frac{U_1}{U_0}\cdot\frac{C^{*\prime}_{fx}}{C_f^{*2}}\right\}\frac{2\nu}{U_0}\right.$$
$$\left. + \left\{\mathcal{B}\,\xi + \frac{1}{2}\left(1-\frac{4}{3}\cdot\frac{U_1}{U_0}\right)\frac{U_1}{U_0}\,\frac{2\nu}{U_0}\cdot\frac{1}{C_f^*}\right\}\frac{U'_{1x}}{U_0}\right],$$

---

* For the values of $\frac{U_1}{U_0}$ and Φ_n to be considered, we find $\mathcal{A} > 0$, $\mathcal{B} < 0$.

i.e., /37

$$C_f^* = 2\Big[\mathcal{A}\,\xi'_x - \frac{\nu}{U_0}\left(\frac{U_1}{U_0}\right)^2\left(1-\frac{2}{3}\frac{U_1}{U_0}\right)\frac{C^{*\prime}_{f\,x}}{C_f^{*2}} + \Big\{\mathcal{B}\,\xi + \frac{2\nu}{U_0}\cdot\frac{U_1}{U_0}\left(1-\frac{U_1}{U_0}\right)\frac{1}{C_f^*}\Big\}\frac{U'_{1x}}{U_0}\Big] \qquad \text{(IIIa)}$$

which will be the form to be used below.

12. Study of the Sublayer Relation and Functional Hypotheses

To solve the problem with respect to the variable x, it is now necessary to solve the system of three simultaneous nonlinear differential equations (I), (II), (III) [or, which comes to the same, resolve the system (Ia), (II), and (IIIa) by introducing the auxiliary function C_f^* over eq.(IV)].

The constants of integration must be fixed by taking the initial conditions into consideration, i.e., conditions fixed by the connectivity in x_j (assumed as given) with the laminar Blasius state.

Unfortunately, it is impossible to directly solve the system of the three equations in question, since we do not know how to write the equation connecting ϵ with U_1 (other than in a differential form).

This difficulty makes us attempt to solve eq.(II) of the sublayer separately, while still producing a class of solutions compatible with eqs.(I) and (III).

This constitutes a detour in approaching our problem, but we will demonstrate that this detour nevertheless will yield information on the mechanism of the phenomena.

Since eq.(II) contains ϵ and U_1 as unknown functions, it is necessary - if this function is considered separately - to conceive a scheme of formation of the sublayer which is realized by the introduction of a second relation between ϵ and U_1, sufficiently simple for permitting a calculation of eq.(II).

Then, the solutions ϵ, U_1 obtained in this manner must be compared with the conditions (I) and (III), for defining the validity of the scheme under consideration.

12.1 Schemes for Causing Rotation of the Sublayer Elements

Actually, we will examine several of such schemes:

First scheme. The sublayer involves the same fluid particles, meaning that it constitutes a stream tube. Then, the relation between ϵ and U_1 is especially simple, being

$$\varepsilon \cdot U_1 = \text{const}\cdot h, \qquad \text{or} \qquad \varepsilon = \frac{h}{U_1} = hU_1^{-1}.$$

The local friction coefficient reads /38

$$C_f^* = 2\frac{\nu}{U_0^2}\cdot\frac{U_1}{\varepsilon} = 2\frac{\nu}{U_0^2}\cdot\frac{U_1^2}{h}.$$

It will be shown below that U_1 is a necessarily descending function of x. This scheme appears incompatible with what is known of the rapid increase in C_f^* in the domain directly downstream of the segment of connectivity x_j with the laminar state preceding the "turbulent" state.

Second scheme. Let us apply the Blasius hypothesis to the sublayer, i.e., let us assume invariance of the pressure in the sublayer. Then, $\frac{\partial P}{\partial Y} = 0$.

Since, the normal pressure σ_y whose expression is

$$\sigma_y = -p + 2\mu\frac{\partial V}{\partial Y} \quad (\mu \text{ is the viscosity coefficient})$$

is invariant in the thickness of the sublayer, it follows that

$$\frac{\partial \sigma_y}{\partial Y} = -\frac{\partial p}{\partial Y} + 2\mu\frac{\partial^2 V}{\partial Y^2} = 0,$$

i.e., $\frac{\partial^2 V}{\partial Y^2} = 0$.

Then, the continuity condition $\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0$ yields

$$\frac{\partial^2 V}{\partial Y^2} = -\frac{\partial^2 U}{\partial x \partial Y} = -\left(\frac{U_1}{\varepsilon}\right)'_x = 0,$$

i.e., $\frac{U_1}{\epsilon} = \text{const} \cdot \frac{1}{h}$ relative to x, which means again that $\epsilon = hU_1$.

Thus, the local coefficient of friction is expressed by

$$C_f^* = \frac{2\nu}{U_0^2}\cdot\frac{1}{h} = \text{const}.$$

Here again we do not have an image permitting consideration of the phenomena occurring in the zone directly downstream of the connectivity segment x_j.

Third scheme. The preceding schemes have proved, with respect to the /39 evolutions U_1 and ϵ, the existence of the laws $\frac{\epsilon'_x}{\epsilon} = -\frac{U'_{1x}}{U_1}$, $\frac{\epsilon'_x}{\epsilon} = \frac{U'_{1x}}{\epsilon}$.

Let us see what happens in the general case $\frac{\epsilon'_x}{\epsilon} = \zeta \cdot \frac{U'_{1x}}{U_1}$ where ζ is an available parameter.

From this it follows that

$$\operatorname{Log} \varepsilon = \operatorname{Log} U_1^{1+\zeta} + \text{const},$$

i.e.,

$$\varepsilon = h U_1^{1+\zeta},$$

where h is a constant satisfying the initial condition. Then, the velocity and rotation gradients have the following expressions:

$$\frac{U_1}{\varepsilon} = \frac{1}{h U_1^{\zeta}}$$

$$\omega \cong -\frac{1}{2} \cdot \frac{U_1}{\varepsilon} = -\frac{1}{2} \cdot \frac{1}{h U_1^{\zeta}}$$

$$\frac{\partial \omega}{\partial x} = -\frac{\zeta}{2} \cdot \frac{U'_{1x}}{h U_1^{1+\zeta}}.$$

Let us write the equilibrium of a fluid element between the actions of pressure, friction, and inertia. We obtain

$$\frac{\partial p}{\partial x} = -\rho U \cdot \frac{\partial U}{\partial x} + 2\mu \cdot \frac{\partial^2 U}{\partial x^2}$$

where

$$U = \frac{U_1}{\varepsilon} Y, \qquad \frac{\partial U}{\partial x} = \left(\frac{U_1}{\varepsilon}\right)'_x \cdot Y = -\frac{\zeta U'_{1x}}{h U_1^{1+\zeta}} = -\frac{\zeta U'_{1x}}{\varepsilon} Y.$$

Finally,

$$\frac{\partial^2 U}{\partial x^2} = \left(\frac{U_1}{\varepsilon}\right)''_{x^2} \cdot Y = -\frac{\zeta}{h} \frac{\partial}{\partial x}\left(\frac{U'_{1x}}{U_1^{1+\zeta}}\right) \cdot Y = -\zeta \frac{\partial}{\partial x}\left(\frac{U'_{1x}}{\varepsilon}\right) \cdot Y;$$

$$\left(\frac{U'_{1x}}{h U_1^{1+\zeta}}\right)'_x = \frac{U''_{1x^2}}{h U_1^{1+\zeta}} - \frac{(1+\zeta) U'^2_{1x}}{h U_1^{2+\zeta}} = \frac{1}{\varepsilon} \left\{ U''_{1x^2} - (1+\zeta) \frac{U'^2_{1x}}{U_1} \right\}.$$

On the average, in the sublayer ϵ, we thus have $\left(\text{since } \frac{1}{\epsilon} \int_0^{\epsilon} Y^2 dY = \frac{\epsilon^2}{3},\ \frac{1}{\epsilon} \int_0^{\epsilon} Y dY = \frac{\epsilon^2}{2}\right)$

$$\frac{1}{\rho} \cdot \frac{\partial p_m}{\partial x} = \zeta \left[\frac{1}{3} U_1 U'_{1x} - \nu \left\{ U''_{1x^2} - (1+\zeta) \frac{U'^2_{1x}}{U_1} \right\}\right].$$

Consequently, ζ appears as characteristic of the evolution of pressure /40 (and of rotations) in the sublayer. We will stipulate that this quantity is characteristic of the mechanism of placing this sublayer in rotation.

It should be noted that, for $\zeta \to 0$, $\dfrac{\partial p_m}{\partial x} \to 0$, this agrees with the observations made with respect to the second scheme.

Use of the relation $\epsilon = hU_1^{1+\zeta}$ for solving the condition (II) relative to the sublayer takes nothing away from the generality of the study since, as is necessary to do, this study is performed in steps: ζ (and h) will be quantities derived step by step in such a manner as to satisfy the other relations (II) and (III) at each step and, at the origin of each of these, also the conditions of connectivity with the preceding step. In fact, a simple change of variables, facilitating the analysis, is involved here.

12.3 Evolution of the Friction Coefficient with the Mean Parameter ζ

It now is necessary, in the case in which the mechanism ζ of induction of rotation in the sublayer varies from step Δx to step Δx, to define the evolutions of thickness ϵ of the sublayer and of the gradient $\dfrac{U_1}{\epsilon}$ in this layer.

Let ζ_j be the mechanism of the first step, with ϵ_j being the thickness of the sublayer at the origin of this step:

$$\varepsilon_j = h_j\, U_{1_j}^{1+\zeta_j} \quad \text{yields} \quad h_j = \frac{\varepsilon_j}{U_{1_j}^{1+\zeta_j}}.$$

Along the step, h_j and ζ_j are constant. Hence,

$$\Delta\,\varepsilon_j = h_j\,(1+\zeta_j)\,U_{1_j}^{\zeta_j}\,\Delta\,U_{1_j} = \frac{\varepsilon_j}{U_{1_j}}(1+\zeta_j)\,\Delta\,U_{1_j}.$$

At the extremity of the step, we thus have

$$\varepsilon_1 = \varepsilon_j + \Delta\,\varepsilon_j = h_1\,U_{1_1}^{1+\zeta_1} = \varepsilon_j\left[1 + \frac{1+\zeta_j}{U_{1_j}}\,\Delta\,U_{1_j}\right],$$

whence

$$h_1 = \frac{\varepsilon_1}{U_1^{1+\zeta_1}} = \varepsilon_j\,\frac{1+(1+\zeta_j)\dfrac{\Delta\,U_{1_j}}{U_{1_j}}}{(U_{1_j}+\Delta\,U_{1_j})^{1+\zeta_j}}.$$

In the second step, the following appears:

$$\Delta \varepsilon_1 = h_1 (1 + \zeta_1)\, U_1^{\zeta_1} \Delta U_1 = \varepsilon_1 \frac{(1 + \zeta_1)\, U_{1_1}^{\zeta_1} \Delta U_{1_1}}{U_{1_1}^{1+\zeta_1}},$$

whence 41

$$\varepsilon_2 = \varepsilon_1 + \Delta \varepsilon_1 = \varepsilon_1 \left[1 + (1 + \zeta_1) \frac{\Delta U_{1_1}}{U_{1_1}} \right] = \varepsilon_j \left(1 + \frac{1 + \zeta_j}{U_1} \Delta U_{1_j} \right) \left(1 + (1 + \zeta_1) \frac{\Delta U_{1_1}}{U_{1_1}} \right)$$

and so on.

The expression of ϵ_n at the n-th step reads

$$\varepsilon_n = \varepsilon_j \prod\nolimits_0^n \left(1 + (1 + \zeta_n) \frac{\Delta U_{1_n}}{U_{1_n}} \right),$$

where $\prod_0^n$ denotes the product sign.

If the quantities ΔU_{1_n} are very small, the principal portion of the product Π will be written as

$$\left[1 + \sum_0^n (1 + \zeta_n) \frac{\Delta U_{1_n}}{U_{1_n}} \right],$$

and, in the form of elementary steps,

$$\varepsilon = \varepsilon_j \left[1 + \int_{u_{1j}}^{u_1} [1 + \zeta (U_1)] \frac{dU_1}{U_1} \right].$$

Taking the mean value of ζ into consideration, this value is removed from under the integral sign so that we can write

$$\varepsilon \cong \varepsilon_j \left[1 + (1 + \zeta) \operatorname{Log} \frac{U_1}{U_{1j}} \right].$$

Simultaneously, the velocity gradient in the sublayer becomes

$$\frac{U_1}{\varepsilon} = \frac{U_1}{\varepsilon_j} \cdot \frac{\dfrac{U_1}{U_{1j}}}{1 + (1 + \zeta) \operatorname{Log} \dfrac{U_1}{U_{1j}}}.$$

Since

$$C_f^* = \frac{\mu \dfrac{\partial U}{\partial Y}}{\dfrac{\rho}{2} U_0^2} = \frac{2\nu}{U_0} \cdot \frac{U_0}{\varepsilon},$$

we have

$$C_f^* = C_{f_j}^* \frac{\dfrac{U_1}{U_{1_j}}}{1 + (1+\zeta)\,\mathrm{Log}\,\dfrac{U_1}{U_{1_j}}},$$

which shows that ζ characterizes also the evolution of the local coefficient of friction.

13. Solutions $U_1(x)$ at Different Values of ζ /42

Let us now return to eq.(II) for rotations in the sublayer, at arbitrary ζ:

$$\frac{U_1}{\varepsilon} = \frac{1}{h\,U_1^{\zeta}}.$$

This equation is written as

$$\frac{U_1}{2}\left(\frac{1}{hU_1^{\zeta}}\right)'_x = \nu\left(\frac{1}{hU_1^{\zeta}}\right)''_{x^2}.$$

Since $\left(\dfrac{1}{hU_1^{\zeta}}\right)'_x = -\dfrac{\zeta U'_{1x}}{hU_1^{\zeta+1}}$, it follows that

$$-\frac{U'_{1x}}{U_1^{\zeta}} = 2\nu\left(-\frac{U'_{1x}}{U_1^{\zeta+1}}\right)'_x.$$

By integration, we obtain

$$\frac{1}{\zeta-1}\cdot\frac{1}{U_1^{\zeta-1}} = -2\nu\frac{U'_{1x}}{U_1^{\zeta+1}} - \text{const.}$$

Let us put

$$\text{Const} = -\frac{1}{\zeta-1}\cdot\frac{1}{\mathfrak{A}^{\zeta-1}},$$

where $\mathfrak{A}$ is the value of U_1 that cancels U'_{1x}.

From this, we derive

$$U'_{1x} = \frac{U_1^{\zeta+1}}{(1-\zeta)\,2\nu}\left[\frac{1}{U_1^{\zeta-1}} - \frac{1}{\mathfrak{A}^{\zeta-1}}\right],$$

i.e., again

$$U'_{1x} = + \frac{1}{(1-\zeta)\,2\nu\,\mathfrak{A}^{\zeta-1}} \cdot U_1^2\,[U_1^{\zeta-1} - \mathfrak{A}^{\zeta-1}],$$

which is a differential equation of the first order with separate variables whose integral solution, from the origin conditions U_{1j}, x_j, reads as follows:

$$\int_{u_{1j}}^{u_1} \frac{dU_1}{U_1^2\,(U_1^{\zeta-1} - \mathfrak{A}^{\zeta-1})} = -\frac{x - x_j}{(1-\zeta)\,2\nu\,\mathfrak{A}^{\zeta-1}}.$$

No matter whether ζ is an integer or a fraction, it is possible to calculate $\int_{u_{1j}}^{u_1}$ in a simple manner. (For a fractional ζ, we will set

$$u = U^{(\zeta-1)\frac{1}{n}},$$

whence /43

$$U_1 = u^{\frac{n}{\zeta-1}}, \qquad U'_{1x} = \frac{n}{\zeta-1} \cdot u^{\frac{n}{\zeta-1}-1} \cdot u'_x, \qquad U_1^2 = u^{\frac{2n}{\zeta-1}}.$$

The quantity n is selected such that $\frac{n}{\zeta - 1}$ is an integer. Decomposition of the quantity under the integral sign will then become possible so that also the integration can be carried out.)

The preceding form will be used for $\zeta > 1$.

For $-1 < \zeta < 1$, we will write

$$U'_{1x} = \frac{1}{(1-\zeta)\,2\nu}\,U_1^{1+\zeta}\,[U_1^{1-\zeta} - \mathfrak{A}^{1-\zeta}],$$

i.e.,

$$\int_{u_{1j}}^{u_1} \frac{dU_1}{U_1^{1+\zeta}\,(U_1^{1-\zeta} - \mathfrak{A}^{1-\zeta})} = \frac{1}{(1-\zeta)\,2\nu}\,(x - x_j).$$

Finally, for $\zeta = 1$ a singular solution is obtained, while the limiting value of $(U_1^{1-\zeta} - \mathfrak{A}^{1-\zeta})\,\frac{1}{1-\zeta}$ must be sought as soon as $\zeta \to 1$ (since this takes the form $\frac{0}{0}$ for $\zeta = 1$).

Since

$$(U_1^{1-\zeta})'_\zeta = -U_1^{1-\zeta}\,\mathrm{Log}\,U_1,$$
$$(\mathfrak{A}^{1-\zeta})'_\zeta = -\mathfrak{A}^{1-\zeta}\,\mathrm{Log}\,\mathfrak{A},$$

making use of a Taylor expansion of $U_1^{1-\zeta} - \mathfrak{A}^{1-\zeta}$ about $\zeta = 1$, it follows that

$$(U_1^{1-\zeta} - \mathfrak{A}^{1-\zeta}) = \lim (U_1^{1-\zeta} - \mathfrak{A}^{1-\zeta})_{\zeta\to 1} - \frac{1-\zeta}{1} \lim (U_1^{1-\zeta}\,\mathrm{Log}\,U_1 - \mathfrak{A}^{1-\zeta}.\,\mathrm{Log}\,\mathfrak{A})_{\zeta\to 1}.$$

Thus,

$$\lim \left(\frac{U_1^{1-\zeta} - \mathfrak{A}^{1-\zeta}}{1-\zeta}\right)_{\zeta\to 1} = -\lim (\mathrm{Log}\,U_1 - \mathrm{Log}\,\mathfrak{A})_{\zeta\to 1} = -\mathrm{Log}\,\frac{U_1}{\mathfrak{A}}.$$

Finally,

$$U'_{1_x} = \frac{1}{2\nu} U_1^2\,\mathrm{Log}\,\frac{U_1}{\mathfrak{A}}\,.$$

In all these cases, it is immediately obvious that the form of evolution /44 of $U_1(x)$ is that figured out above (Fig.7). The curve consists of three branches. Only one does not contain ∞ for U_1, which is the only one to be taken into consideration. In this branch, U'_{1_x} is constantly negative while U_1 evolves by decreasing between two asymptotes, $\mathfrak{A}$ for $x = -\infty$ and zero for $x = +\infty$.

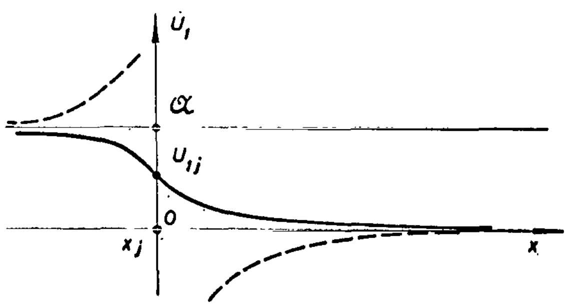

Fig.7

It will be noted that the presence of the factor $\frac{1}{2\nu}$ in front of $x = x_j$, in view of the very low value of its inverse (29×10^{-6} for 2ν in the usual cases), indicates how minimal must be the path $x - x_j$ necessary for having U_1 pass from U_{1_j} (which will be of the order of 0.65 U_0) to values directly adjacent to the asymptotic values*. This simply signifies that the initiations of rotation analyzed by eq.(II) rapidly reach their state of equilibrium and that viscosity effects of an entirely different order of magnitude are required for appreciably decelerating this process.

14. Necessity of a Finite Lower Limit for the Decrease in U_1

Since U_1, starting from U_{1_j} which is the initial connectivity value between the "stationary turbulent" state and the laminar state, rapidly approaches its asymptotic limit, it will be found that U'_{1_x} vanishes extremely rapidly: Thus, eqs.(Ia) and (IIIa) assume the following reduced forms:

* An explanation for a much slower evolution will be given in the second part (nonstationary study).

$$\xi'_x - \frac{\sum_n \frac{1}{n^2}}{2\pi^2}\xi^2\xi'''_{x^3} \approx \frac{\nu}{U_1}\cdot\frac{\pi^2}{2}\cdot\frac{B}{A}\cdot\frac{1}{\xi}, \tag{Ib}$$

$$C_f \cong 2\left[\mathcal{A}\,\xi'_x - \frac{\nu}{U_0}\left(\frac{U_1}{U_0}\right)^2\left(1-\frac{2}{3}\cdot\frac{U_1}{U_0}\right)\frac{C^{*\prime}{}_{/x}}{C^{*2}{}_f}\right]. \tag{IIIb}$$

If U_1 would tend to zero as indicated by the preceding solutions, it would be necessary that at least ξ increase indefinitely, for values $x - x_j$ that are still extremely small*.

This phenomenon might possibly be compatible with flows in the state of separation but certainly cannot be compatible with normal flows. Thus, it is necessary that U_1 find a finite lower limit for its decrease.

However, this would mean that eq.(II) cannot be reduced to the excessively simplified form that we have used by considering, in the rotation, only the term $\omega_i = -\frac{1}{2}\cdot\frac{U_1}{\epsilon}$. The functioning of the sublayer cannot be reduced to a one-dimensional state. The normal velocity component V cannot be neglected when the contraction $\epsilon(x)$ is extremely abrupt.

In that case, a term complementary to those taken into consideration is in existence. Let $\Delta(U_1, x)$ be this term such that /45

$$\omega_i = -\frac{1}{2}\left[\frac{U_1}{\epsilon} - \Delta\right],$$

with ϵ retaining its preceding expression

$$\epsilon = hU_1^{1+\zeta}$$

(where h and ζ vary in principle from step to step).

This makes it necessary to return to the preceding expansions and to define the conditions which Δ must satisfy so that U_1 will find a nonzero asymptotic

* A numerical calculation shows that, if U_{1_j} is not infinitely close to the asymptotic limit, this limit is practically reached for paths $x - x_j$ of $\left(\frac{1}{1000}\right)$ to $\left(\frac{2}{1000}\right)$ of a millimeter. In fact, it is impossible to furnish a simultaneous explanation for the abrupt increase in ξ and C_f in the zone x close to x_j except by assuming that U_{1_j} differs greatly from the asymptotic limit; according to the configuration of the Blasius field and since the boundary of the sublayer is located where U'_Y ceases being constant, a value U_{1_j} of the order of 0.65 to 0.68 U_0 corresponds to this.

lower limit X.

15. Solutions $U_1(x)$ with Complementary Term* in the Sublayer Equation

Here, we thus have

$$\varepsilon = hU_1^{1+\zeta}, \qquad \omega_i = -\frac{1}{2}\left[\frac{1}{hU_1^{\zeta}} - \Delta\right]$$

and

$$\omega'_{i_x} = -\frac{1}{2}\left[-\frac{\zeta\, U'_{1_x}}{hU_1^{1+\zeta}} - \Delta'_{U_1} \cdot U'_{1_x}\right] = \frac{1}{2} \cdot \frac{\zeta}{hU_1^{1+\zeta}}\left[1 + \frac{h}{\zeta}\Delta'_{U_1} \cdot U_1^{(1+\zeta)}\right] U'_{1_x}.$$

The fundamental equation of the sublayer

$$\frac{U_1}{2}\,\omega'_{i_x} = \nu\,\omega''_{i_x},$$

on multiplying by 2, becomes

$$\frac{U_1}{2}\,U'_{1_x}\left[\frac{\zeta}{hU_1^{1+\zeta}} + \Delta'_{U_1}\right] = \nu\left(\left[\frac{\zeta}{hU_1^{1+\zeta}} + \Delta'_{U_1}\right] U'_{1_x}\right)'_x,$$

or

$$U'_{1_x}\left[\frac{\zeta}{hU_1^{\zeta}} + U_1\,\Delta'_{U_1}\right] = 2\,\nu\left(\left[\frac{\zeta}{hU_1^{1+\zeta}} + \Delta'_{U_1}\right] U'_{1_x}\right)'_x,$$

so that

$$\frac{\zeta}{1-\zeta} \cdot \frac{1}{hU_1^{\zeta-1}} + \int U_1 \cdot \Delta'_{U_1}\, dU_1 = 2\,\nu\left(\frac{\zeta}{hU_1^{1+\zeta}} + \Delta'_{U_1}\right) U'_{1_x} + \text{const.}$$

So as to have a root X appear in the denominator of the term in U'_{1x} of the equation written by separating the variables, it is necessary, with /46

$$\frac{\zeta}{h(1-\zeta)}\left[U_1^{1-\zeta} - \mathcal{X}^{1-\zeta}\right] + \int U_1 \cdot \Delta'_{U_1}\, dU_1 = 2\,\nu\left(\frac{\zeta}{hU_1^{1+\zeta}} + \Delta'_{u_1}\right) U'_{1_x}$$

or

$$\frac{\zeta}{h(1-\zeta)}\left[1 - \frac{\mathcal{X}^{1-\zeta}}{U_1^{1-\zeta}}\right] + \frac{1}{U_1^{1-\zeta}}\int U_1 \cdot \Delta'_{u_1}\, dU_1 = 2\,\nu\,\frac{\zeta}{hU_1^{2}}\left(1 + \frac{h}{\zeta}U_1^{1+\zeta} \cdot \Delta'_{U_1}\right) U'_{1_x}$$

---

* Another possibility was also investigated, in which ε would not be able to decrease indefinitely (its minimal dimension could be of the order of the wall roughnesses). It has been proved that such a consideration does not permit the appearance of a lower nonzero limit for U_1 and a decrease of the friction coefficient at increasing $x - x_j$. Therefore, we cannot expect to find here an explanation for the encountered difficulty.

that

$$1 + \frac{h}{\zeta} U_1^{1+\zeta} \cdot \Delta'_{U_1} = \frac{U_1}{U_1 - X}.$$

From this, it follows that

$$\Delta'_{U_1} = \frac{\zeta}{h U_1^{1+\zeta}} \left[-1 + \frac{U_1}{U_1 - X} \right] = \frac{\zeta}{h U_1^{1+\zeta}} \cdot \frac{X}{U_1 - X}.$$

Hence,

$$\Delta = \frac{\zeta X}{h} \int \frac{dU_1}{U_1^{1+\zeta} (U_1 - X)} + \text{const} C_2$$

and

$$\int U_1 \cdot \Delta'_{U_1} \cdot dU_1 = \frac{\zeta X}{h} \int \frac{dU_1}{U_1^{\zeta} (U_1 - X)} + \text{const} C_1.$$

15.1 Integration of the Sublayer Equation with Complementary Term

Let us return to the fundamental equation which now becomes

$$\frac{\zeta}{h(1-\zeta)} \left[1 - \frac{\mathcal{U}^{1-\zeta}}{U_1^{1-\zeta}} \right] + \frac{1}{U_1^{1-\zeta}} \int U_1 \cdot \Delta'_{U_1} \, dU_1 = 2\nu \frac{\zeta}{h U_1} \cdot \frac{U'_{1_x}}{U_1 - X}$$

or

$$\frac{1}{1-\zeta} \left[U_1^{1-\zeta} - \mathcal{U}^{1-\zeta} \right] + \frac{h}{\zeta} \int U_1 \cdot \Delta'_{U_1} \, dU_1 = 2\nu \cdot \frac{U'_{1_x}}{U_1^{\zeta} (U_1 - X)}.$$

We will treat (to recapitulate) the case $\zeta = 1$ and the case $\zeta = 0$.

Case $\zeta = 1$:

$$\Delta = \frac{X}{h} \int \frac{dU_1}{U_1^2 (U - X)}, \qquad \int U_1 \cdot \Delta'_{U_1} \, dU_1 = \frac{X}{h} \cdot \frac{1}{X} \operatorname{Log} \frac{U_1 - X}{U_1} + \text{const.}$$

Let us calculate

$$\lim \left[\left(\frac{U_1^{1-\zeta}}{\mathcal{U}^{1-\zeta}} - 1 \right) \mathcal{U}^{1-\zeta} \right] \left(\frac{1}{1-\zeta} \right)$$

when $\zeta \to 1$, by putting

$$f = \left[1 - \left(\frac{\mathcal{U}}{U_1} \right)^{\varepsilon} \right],$$

whence

$$f'_{\varepsilon} = - \operatorname{Log} \frac{\mathcal{U}}{U_1} \left(\frac{\mathcal{U}}{U_1} \right)^{\varepsilon}, \qquad f''_{\varepsilon^2} = \left(\operatorname{Log} \frac{\mathcal{U}}{U_1} \right)^2 \cdot \left(\frac{\mathcal{U}}{U_1} \right)^{\varepsilon} \ldots.$$

On expanding this in a Taylor series, we obtain /47

$$f(\varepsilon) = f(0) + \frac{\varepsilon}{1} f'_{\varepsilon}(0) + \frac{\varepsilon^2}{1 \cdot 2} f''(0) \dots$$

where

$$f(0) = 0, \qquad f'(0) = -\mathrm{Log}\frac{\mathfrak{A}}{U_1}, \qquad f''(0) = \left(\mathrm{Log}\frac{\mathfrak{A}}{U_1}\right)^2.$$

Thus,

$$\frac{f(\varepsilon)}{\varepsilon} = f'(0) + \frac{\varepsilon}{1 \cdot 2} f''(0) \dots = -\mathrm{Log}\frac{\mathfrak{A}}{U_1}$$

when $\varepsilon \to 0$.

Consequently, as $\zeta \to 1$, the equation is written in the form

$$-\mathrm{Log}\frac{\mathfrak{A}}{U_1} + \mathrm{Log}\frac{U_1 - X}{U_1} + \mathrm{const}\, C_1 = 2\nu \frac{U'_{1x}}{U_1(U_1 - X)},$$

whence

$$\frac{U'_{1x}}{U_1(U_1 - X)\,[\mathrm{Log}(U_1 - X) - \mathrm{Log}\,\mathfrak{A} + C_1]} = \frac{1}{2\nu}.$$

We will now put $C_1 = \mathrm{Log}\dfrac{\mathfrak{A}}{\mathfrak{A} - X}$ which, as above, will permit obtaining $U'_{1x} \to 0$ as $U_1 \to \mathfrak{A}$; in fact, we then have

$$U'_{1x} = \frac{1}{2\nu} U_1(U_1 - X)\,\mathrm{Log}\frac{U_1 - X}{\mathfrak{A} - X}.$$

Calculation of the integral can be carried out by graphical means.

For Δ, we obtain

$$\Delta = \frac{X}{h}\left[-\frac{1}{X}\int\frac{dU_1}{U_1^2} \pm \frac{1}{X^2}\left(\int\frac{dU_1}{U_1 - X} - \int\frac{dU_1}{U_1}\right)\right] = \frac{1}{h}\left[\frac{1}{U_1} + \frac{1}{X}\mathrm{Log}\frac{U_1 - X}{U_1} + \mathrm{const}\, C_2\right],$$

$$h\Delta = \left[\frac{1}{U_1} + \frac{1}{X}\mathrm{Log}\left(1 - \frac{X}{U_1}\right)\right] + \mathrm{const}\, C_2$$

and

$$\omega_l = -\frac{1}{2h}\left[\frac{1}{U_1} - h\Delta\right] = -\frac{1}{2h}\left[\frac{1}{U_1} - \left\{\frac{1}{U_1} + \frac{1}{X}\mathrm{Log}\frac{U_1 - X}{U_1}\right\} + C_2\right]$$

$$= \frac{1}{2h}\left[\frac{1}{X}\mathrm{Log}\frac{U_1 - X}{U_1}\right] + C_2.$$

Let us now investigate the case $\zeta = 0$.

Case $\zeta = 0$.

In this case, $\Delta \to 0$ but

$$\frac{h}{\zeta}\Delta = X\int \frac{dU_1}{U_1(U_1 - X)} + \text{const},$$

i.e., 48

$$h\frac{\Delta}{\zeta} = \operatorname{Log}\frac{U_1 - X}{U_1} + \text{const}$$

and $\Delta \to 0$ if the constant is zero.

Similarly,

$$\frac{h}{\zeta}\int \Delta'_{u_1} U_1\, dU_1 = X\int \frac{dU_1}{U_1 - X} + \text{const} = X \operatorname{Log}(U_1 - X) + \text{const}.$$

The fundamental equation is then written as

$$(U_1 - \mathfrak{A}) + X\operatorname{Log}(U_1 - X) + \text{const} = 2\nu\frac{U'_{1x}}{U_1 - X}$$

or

$$\frac{U'_{1x}}{(U_1 - X)\left[U_1 - \mathfrak{A} + X\operatorname{Log}(U_1 - X) + \text{const}\right]} = \frac{1}{2\nu}.$$

We put

$$\text{const} = -X\operatorname{Log}(\mathfrak{A} - X)$$

so that $U'_{1x} \to 0$ as soon as $U_1 \to \mathfrak{A}$:

$$U'_{1x} = \frac{1}{2\nu}(U_1 - X)\left[U_1 - \mathfrak{A} + X\operatorname{Log}\frac{U_1 - X}{\mathfrak{A} - X}\right]$$

and, finally,

$$\frac{U'_{1x}}{U_0} = -\frac{U_0}{2\nu}\cdot\frac{U_1 - X}{U_0}\left[\frac{\mathfrak{A} - U_1}{U_0} + \frac{X}{U_0}\operatorname{Log}\frac{\mathfrak{A} - X}{U_1 - X}\right],$$

a form to which we will refer from now on. This expression yields

$$x - x_j = -\frac{2\nu}{U_0}\int_{U_{1j}}^{U_1}\frac{dU_1}{\frac{U_1 - X}{U_0}\left[\frac{\mathfrak{A} - U_1}{U_0} + \frac{X}{U_0}\operatorname{Log}\frac{\mathfrak{A} - X}{U_1 - X}\right]}.$$

From this, $U''_{1_{x^2}}$ and $U'''_{1_{x^3}}$ is readily derived as a function of U'_{1_x} and U_{1_x} (and of U_1 alone, since U'_{1_x} is defined in U_1):

$$U''_{1_{x^2}} = \frac{U'_{1_x}}{2\nu}\left(2\,U_1 - \mathfrak{A} + X\,\mathrm{Log}\,\frac{U_1 - X}{\mathfrak{A} - X}\right)$$

$$= \frac{U_1 - X}{4\nu^2}\left[U_1 - \mathfrak{A} + X\,\mathrm{Log}\,\frac{U_1 - X}{\mathfrak{A} - X}\right]\left[2\,U_1 - \mathfrak{A} + X\,\mathrm{Log}\,\frac{U_1 - X}{\mathfrak{A} - X}\right]$$

and /49

$$U'''_{1_{x^3}} = \frac{U'_{1_x}}{4\nu^2}\left[\left\{\left[U_1 - \mathfrak{A} + X\,\mathrm{Log}\,\frac{U_1 - X}{\mathfrak{A} - X}\right]\left(2\,U_1 - \mathfrak{A} + X\,\mathrm{Log}\,\frac{U_1 - X}{\mathfrak{A} - X}\right)\right\}\right.$$
$$\left. + (U_1)\left(2\,U - \mathfrak{A} + X\,\mathrm{Log}\,\frac{U_1 - X}{\mathfrak{A} - X}\right) + (2\,U_1 - X)\left(U_1 - \mathfrak{A} + X\,\mathrm{Log}\,\frac{U_1 - X}{\mathfrak{A} - X}\right)\right]$$

[which will permit introducing the term $\dfrac{U'''_{1_{x^3}}}{U'_x} = K(U_1)$ into the calculations relative to eq.(I)].

The form of evolution of $U_1(x)$, with the complementary term Δ, is that given below (irrespective of the value of ζ); Fig.8.

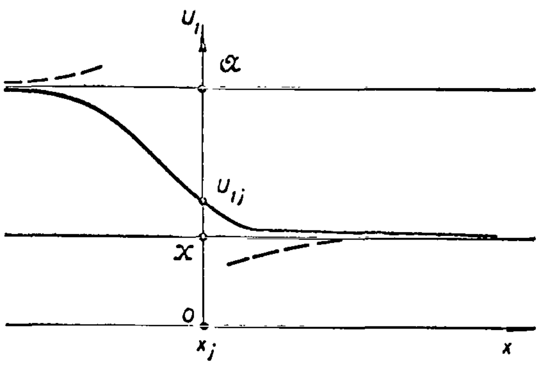

Fig.8

In fact, as mentioned above, the length of the path $x - x_j$ required for a decrease in U_1 from its initial value to values extremely close to the asymptotic limit X, is so short that it is of no importance for the numerical result to go into details on the step-by-step evolution phenomenon of ζ. In practice, it would be possible to use the numerical solution furnished by the case $\zeta = 0$ and to retain the existence of the other solutions only for a more physical analysis of the phenomena of initiation of rotation of the particles in the sublayer.

In particular, it will be noted that these particles are directly dependent on the viscosity factor $\dfrac{U_0}{2\nu}$.

[We are giving here tabulated values of the calculation for two cases of evolution of $U_1(x)$ between $U_{1_f} = 0.66\ U_0$ and $X = 0.45\ U_0$ for $\dfrac{\mathfrak{A}}{U_0} = 1$.]

$\frac{U_1}{U_0}$	0.65	0.60	0.55	0.50	0.47^5	0.45	
$U_0 \frac{x - x_j}{\nu}$	0	0.646	1.396	2.476	3.346	∞	Case $\zeta = 0$
$U_0 \frac{(x - x_j)}{\nu}$	0	0.810	1.778	3.154	4.250	∞	Case $\zeta = 1$

16. Determination of the Constant $\mathfrak{A}$ /50

For $x = -\infty$, $\mathfrak{A}$ represents the asymptotic value of U_1 (in all above-defined cases, we were careful to determine the constants in such a manner that $U_1 \to \mathfrak{A}$ when $U'_{1_x} \to 0$).

According to what has been shown above, U_1 can never be superior to its initial value U_{1_j} of the connectivity segment between the laminar and "stationary turbulent" states.

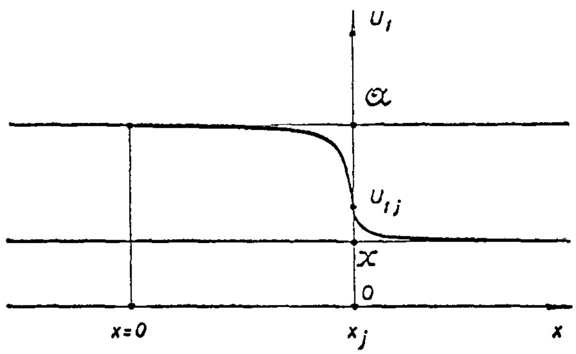

Fig.9

Thus, $\mathfrak{A}$ will be the fictitious value of U_1 at infinity upstream, i.e., on a plate extending to infinity upstream for which the "stationary turbulent" flow will start from the leading edge (Fig.9). In fact, the same reason which caused us to state that U_1 practically reaches its lower asymptotic limit X for a minimum path $x - x_j > 0$ is applicable here. In the fictitious flow, which is "turbulent" from the leading edge of the plate, U_1 will retain a value practically equal to $\mathfrak{A}$ from the leading edge (even if this is at a short distance x_j from the critical segment) up to the direct vicinity of x_j.

However, at the leading edge itself, the boundary sublayer consists of a first particle starting to rotate and then rolling along the wall; this layer is such - or at least we must conceive it as such - that it alone constitutes the sublayer of constant velocity gradient U'_Y, ensuring a cancellation of the velocity along the wall and a connection with the velocity U_0 of the external flow (Fig.10). The actual boundary layer /51 does not yet exist since a single layer of particles forms the entire boundary layer. From now on, we must set $\mathfrak{A} = U_0$.

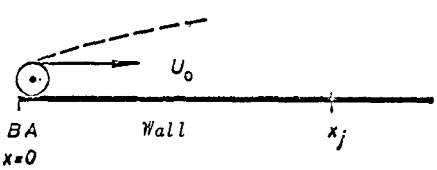

Fig.10

Evidently, this reasoning on a fictitious state is quite precarious. Nevertheless, we are convinced that - until proved differently - it is entirely logical.

17. Return to the Relations of the Problem in x

The relations to be used here are finally as follows:

A relation (I) of the base of the actual boundary layer, written in the form

$$\xi\,\xi'_x - \frac{1}{2\pi^2}\cdot\sum\left(\frac{1}{n^2}\right)\cdot\xi^3\,\xi'''_{x^3} = \frac{\nu}{2\,U_0}\cdot\frac{\pi^2\frac{B}{A}}{\frac{U_1}{U_0}} - \frac{\xi^2}{1-\frac{U_1}{U_0}}\left[\frac{U'_{1x}}{U_0} - \frac{1}{2\pi^2}\sum_n\left(\frac{1}{n^2}\right)\xi^2\cdot\frac{U'''_{1x^3}}{U_0}\right]. \qquad \text{(Ib)}$$

A relation (III) for variations in the losses of momentum, written for the local friction coefficient

$$C^*_f + 2\frac{\nu}{U_0}\left(\frac{U_1}{U_0}\right)^2\left(1-\frac{2}{3}\cdot\frac{U_1}{U_0}\right)\frac{C^{*\prime}_{fx}}{C^{*2}_f} = 2\left[\mathcal{A}\,\xi'_x + \left\{\mathcal{B}\,\xi + \frac{2\nu}{U_0}\cdot\frac{U_1}{U_0}\left(1-\frac{U_1}{U_0}\right)\frac{1}{C^*_f}\right\}\frac{U'_{1x}}{U_0}\right]. \qquad \text{(IIIb)}$$

A relation (II) for the base of the sublayer

$$\frac{U'_{1x}}{U_0} \cong -\frac{U_0}{2\,\nu}\cdot\frac{U_1 - X}{U_0}\left[\frac{U_0 - U_1}{U_0} + \frac{X}{U_0}\,\mathrm{Log}\,\frac{U_0 - X}{U_1 - X}\right]. \qquad \text{(IIb)}$$

Finally, this must be supplemented by

$$\varepsilon = \frac{U_1}{U_0}\cdot\frac{2\nu}{U_0}\cdot\frac{1}{C^*_f}.$$

18. Integration by Parts; Development of the Quantities

It should now be possible to make a step-by-step calculation of $\xi(x)$ and $C^*_f(x)$ from eqs.(I) and (III) since $U_1(x)$ can be determined by different means.

However, one difficulty arises here: The demonstrated solution $U_1(x)$ shows an initially extremely rapid evolution of U_1 with x, meaning that $U'_{1x} < 0$ is highly important. Here, we generally are no longer within the scope of our basic approximations with respect to eq.(I).

18.1 Case in which Integration is Possible /52

We will return to this case if the rotational term varies little, which happens in the two following cases:

The first case is that in which U_1 has reached the neighborhood of its asymptotic value X and in which the values of U'_{1x}, U''_{1x^2}, ... in eq.(I) have become very low such that the equation is reduced to the following form:

$$\xi\,\xi'_x - \frac{1}{2\pi^2}\Sigma\left(\frac{1}{n^2}\right)\cdot\xi^2\,\xi'''_{x^3} \cong \frac{\nu}{2U_0}\cdot\frac{\pi^2\frac{B}{A}}{\frac{X}{U_0}}. \tag{Ic}$$

Simultaneously, eq.(III) becomes

$$C^*_f + \frac{2\nu}{U_0}\left(\frac{X}{U_0}\right)^2\left(1-\frac{2}{3}\frac{X}{U_0}\right)\frac{C^{*\prime}{}_{fx}}{C_f^{*2}} = 2\,\mathcal{A}\,\xi'_x$$

$$\mathcal{A}\left(\frac{U_1}{U_0}\right)\text{ taking the value }\overline{\mathcal{A}} = \mathcal{A}\left(\frac{X}{U_0}\right). \tag{IIIc}$$

It is here a question of a general case (independent of the point x_j where initiation of the "turbulent" state takes place) which, however, does not define the evolution of ξ (and of C_f^*) in the narrow domain in which U'_{1x} is important.

The second case of possible solution is that in which ξ_j is sufficiently small for having - even if U'_{1x} is large - ϵ'_x, ξ'_x and V remain low and in conformity with the approximations made. Consequently, it is necessary that the Reynolds number $\mathfrak{R}_j = \frac{U_0 x_j}{\nu}$, characteristic of the segment x_j, will be low which would mean that ξ_j is very low.

Conversely, the restriction that ϵ'_x, ξ'_x, V be small will still be respected as long as it is a question of studying the second domain from which eq.(Ic) is derived and for which $U'_{1x} \simeq 0$, without restriction of the Reynolds number.

A method for integrating this equation and the resultant solutions will be demonstrated later in the text.

18.2 Solution of the Second Case

In this Section, we will restrict our investigation to the case of very low $\mathfrak{R}_j$, so as to study the problem in its simplest possible form.

According to a previous statement (Chapt.I, Sect.6.1), the terms in ξ'''_{x^3},

U'''_{1x^3} $\left[\text{originating from } \left(\frac{\bar{U}}{\xi}\right)'''_{x^3} \text{ of the equations of rotation}\right]$ are connected with the presence of a residual deviation $\Delta u(\xi)$ existing at the level $y = \xi$ between the calculated field $U(\xi)$ and U_0. As long as ξ is sufficiently low, the term /53 $\xi^3\xi'''_{x^3}$ will be small with respect to $\xi\xi'_x$. This will also be the case when $\Re_j$ is small (with ξ_j being very small). In that case, we can neglect the term ξ'''_{x^3} in eq.(Ib) which, at the increments Δ, will be written in the form

$$\Delta\xi = \frac{\nu}{2U_0} \cdot \frac{\pi^2 \frac{B}{A}}{\frac{U_1}{U_0}} \cdot \frac{\Delta x}{\xi} - \frac{\xi}{1 - \frac{U_1}{U_0}} \cdot \frac{\Delta U_1}{U_0}$$

with

$$\frac{\Delta U_1}{U_0} = -\frac{U_0}{2\nu} \cdot \frac{U_1 - X}{U_0} \left[\frac{U_0 - U_1}{U_0} + \frac{X}{U_0} \operatorname{Log} \frac{U_0 - X}{U_1 - X}\right] \Delta x \quad \text{(solution to } \zeta = 0\text{)}$$

and

$$\frac{\Delta C^*_l}{C^{*2}_l} = \frac{U_0}{\nu} \cdot \frac{1}{\left(\frac{U_1}{U_0}\right)^2 \left(1 - \frac{2}{3} \cdot \frac{U_1}{U_0}\right)} \left[\mathcal{A}\,\Delta\xi + \left(\frac{2\nu}{U_0} \cdot \frac{U_1}{U_0} \cdot \frac{1 - \frac{U_1}{U_0}}{C^*_l} + \mathcal{B}\,\xi\right) \frac{\Delta U_1}{U_0} - C^*_l \frac{\Delta x}{2}\right].$$

[Let us recall that $\mathcal{A}$ and $\mathcal{B}$ are functions of $\frac{U_1}{U_0}$, with $\mathcal{A}$ being positive and $\mathcal{B}$ being negative (see Sect.11.1).]

The first step has x_j as origin, where the field ceases being laminar.

The Blasius field is such that U practically reaches the value U_0 of the external flow for $Y = \delta_L \cong 5.5\sqrt{\frac{\nu x}{U_0}}$.

As already mentioned, if the last laminar segment x_j is the first segment in which the "stationary turbulent" state begins, then the limit of the sublayer of this segment is marked by the level Y where the gradient U'_y ceases to be constant. Thus, we obtain

$$\frac{\epsilon_j}{\delta_j} \cong 0.365, \qquad U_{1j} \cong 0.65\, U_0,$$

whence

$$\xi_j = \delta_j\left(1 - \frac{\epsilon_j}{\delta_j}\right) = 5.5\,\sigma\sqrt{\frac{\nu x_j}{U_0}} \quad \text{(where } \sigma = 0.635\text{)}.$$

The laminar local friction coefficient, according to Blasius, is expressed by

$$C^*_{lj} = \frac{0.66^5}{\sqrt{\mathcal{R}_j}} = \frac{0.66^5}{\sqrt{\frac{U_0 x_j}{\nu}}}.$$

It will then be noted that

$$\xi_j C^*_{fj} = 3.75 \cdot 0.66^{6} \frac{\nu}{U_0} = 2.485 \cdot \frac{\nu}{U_0}$$

is independent of x_j.

Let us take U_1 as independent variable. Then, to the successive $\Delta U_1 < 0$, /54
starting from the first step, there will first correspond very small Δx $\left(\text{factor } \frac{U_0}{2\nu} \text{ of } \frac{1}{U'_{1x}}\right)$ and $\Delta\xi$ such that

$$\Delta \xi \cong -\frac{\xi}{1-\frac{U_1}{U_0}} \cdot \frac{\Delta U_1}{U_0} > 0.$$

Here, ξ increases rapidly.

Since the term $\frac{2\nu}{U_0} \cdot \frac{U_1}{U_0} \cdot \frac{1-\frac{U_1}{U_0}}{C_f^*}$ is weak with respect to $\left(\frac{\mathcal{A}}{1-\frac{U_1}{U_0}} - \mathcal{B}\right)$, the variations in the coefficient of friction ΔC_f^* are larger than zero, and C_f^* increases rapidly. Then, for the same steps Δx, the terms $\Delta U_1 < 0$ will decrease as soon as U_1 approaches its asymptotic value X.

In $\Delta\xi$, the term $\frac{\nu}{2U_0} \cdot \frac{\pi^2 \frac{B}{A}}{\frac{U_1}{U_0}} \cdot \frac{\Delta x}{\xi}$ becomes the principal term, which means that ξ remains a (slowly) ascending function of x.

So far as the ΔC_f^* are concerned, they will first weaken in the same manner.

As soon as the remoteness $x - x_j$ is sufficient and as soon as U'_{1x} has simultaneously become sufficiently minimal, we will have $\Delta\bar{\xi} \cong \frac{\nu}{2U_0} \cdot \frac{\pi^2 \frac{B}{A}}{\frac{U^1}{U_0}} - \frac{\Delta x}{\xi}$ (with U_1 taking a value very close to X)*.

Simultaneously,

$$\frac{\Delta \bar{C}_f^*}{\bar{C}_f^{*2}} \cong \frac{U_0}{\nu} \cdot \frac{1}{\left(\frac{X}{U_0}\right)^2 \left(1 - \frac{2}{3} \cdot \frac{X}{U_0}\right)} [\bar{\mathcal{A}} \cdot \Delta \bar{\xi} - \frac{1}{2} \bar{C}_f^* \cdot \Delta x].$$

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* To distinguish more readily terms referring to the case in which $U'_{1x} \cong 0$, we will use the notations $\bar{C}_f$ and $\bar{\xi}$ with vinculi. At the same time, $U_1 \to X$.

This means that $\Delta\overline{C}_f^*$ will be zero at $\frac{\Delta\overline{\xi}}{\Delta x} = \frac{\overline{C}_f^*}{2\mathcal{A}}$, i.e., as soon as the value taken by $\overline{C}_f^*$, on substituting $\frac{\Delta\overline{\xi}}{\Delta x}$ by its above value, will become

$$\overline{C}_{f_j}^* = \overline{\mathcal{A}}\,\frac{\nu}{U_0}\cdot\frac{\pi^2\frac{B}{A}}{\frac{X}{U_0}}\cdot\frac{1}{\overline{\xi}_j}.$$

Thus, C_f^* passes through a maximum as soon as the slope ξ'_x is sufficiently shallow: Starting from the corresponding point, C_f^* and ξ will continue to develop in accordance with the reduced equations established in this manner, corresponding to $U_1 = X$, $U'_{1x} = 0$.

The relation $\overline{\xi}'_x \cong \frac{\nu}{2U_0}\cdot\frac{\pi^2\frac{B}{A}}{\frac{X}{U_0}}\cdot\frac{1}{\xi}$ is easy to integrate, yielding /55

$$(\overline{\xi}^2 - \overline{\xi}_j^2) = \frac{\nu}{U_0}\cdot\frac{\pi^2\frac{B}{A}}{\frac{X}{U_0}}\,[x - \bar{x}_j].$$

The quantities $\overline{\xi}_j$, $\overline{x}_j$ are the original values, i.e. those corresponding to the conditions where $\overline{C}_f^*$ passes through its maximum (or very close to it); see Fig.11.

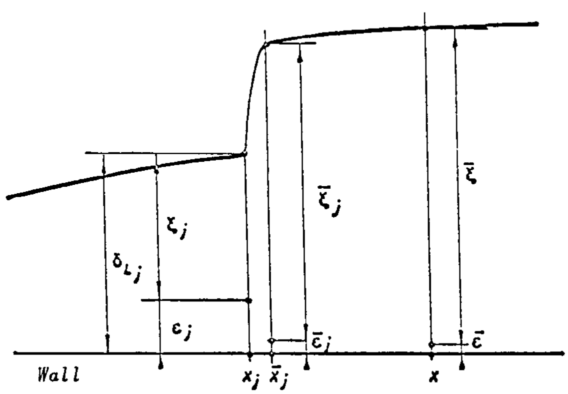

Fig.11

It is already known that $\overline{x}_j \cong x_j$.

Let us attempt to determine $\overline{\xi}_j$:

In the domain $x_j < x < \overline{x}_j$, we have seen that - in principle - the step-by-step variation $\Delta\xi$ resulted from the term in $\Delta U_1 < 0$ by

$$\Delta\xi \cong -\frac{\xi}{1-\frac{U_1}{U_0}}\cdot\frac{\Delta U_1}{U_0}.$$

For ΔU_1, extending from U_{1_j} which is the initial value in the segment of connectivity with the laminar state up to X which is the final value, we thus have

$$\frac{\Delta\xi}{\xi} \simeq \frac{\Delta U_1}{U_0 - U_1}.$$

By integration, we obtain

$$\mathrm{Log}\,\frac{\xi}{\xi_j} \cong \mathrm{Log}\,\frac{U_0 - U_1}{U_0 - U_{1j}},$$

meaning that here /56

$$\frac{\bar{\xi}_j}{\xi_j} = \frac{1 - \frac{X}{U_0}}{1 - \frac{U_{1j}}{U_0}}.$$

Since ξ_j is connected with the thickness δ_L of the laminar Blasius layer in the segment x_j by

$$\xi_j = \sigma\,\delta_{Lj} \cong 5.5\,\sigma\sqrt{\frac{\nu x_j}{U_0}} \qquad (\sigma \cong 0.635)$$

we obtain

$$\bar{\xi}_j = \sigma \cdot 5.5 \cdot \sqrt{\frac{\nu x_j}{U_0}} \cdot \frac{1 - \frac{X}{U_0}}{1 - \frac{U_{1j}}{U_0}}$$

and, somewhat farther on, allowing for the factor in $\frac{\nu}{U_0} \cdot \frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}}$ which had been neglected about x_j (as well as $U_1 \neq X$),

$$\bar{\xi}^2 = \sigma^2 \cdot 5.5^2 \cdot \frac{\nu x_j}{U_0} \cdot \left(\frac{1 - \frac{X}{U_0}}{1 - \frac{U_0}{U_{1j}}}\right)^2 + \frac{\nu}{U_0} \cdot \frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}} (x - x_j)$$

$$= \frac{\nu}{U_0} \cdot \frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}} x + x_j \frac{\nu}{U_0} \left[\sigma^2 \cdot 5.5^2 \left(\frac{1 - \frac{X}{U_0}}{1 - \frac{U_0}{U_{1j}}}\right)^2 - \frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}} \right].$$

It will be noted that the restrictions made for the use of the integration by parts are applicable only to eq.(I) in ξ. Thus, they affect only the law $\xi(x)$. The conclusions drawn, specifically the presence of a maximum for $C_f^*(\bar{x}_j)$, remain valid as does also the corresponding relation

$$\bar{C}_f^* = 2\,\bar{\mathcal{A}} . \bar{\xi}'_x \qquad \left[\bar{\mathcal{A}} = \mathcal{A}\left(\frac{U_1}{U_0}\right) \quad \text{for} \quad U_1 \to X\right],$$

since this is directly derived from eq.(III).

It should be noted also that $\bar{\bar{\xi}}$ would become independent of x_j and would be identified with a unique solution $\bar{\bar{\xi}}(x)$ if the bracket before $x_j \dfrac{\nu}{U_0}$ were zero. At the same time, $\bar{C}^*_{f_j}$ would be identified with a value $\bar{C}^*_f(x)$, independent of x_j. It will be demonstrated below that the latter is a general fact (not connected /57 with small values of $\mathfrak{R}_j$) and that it agrees with practical experimental results.

The first fact [uniqueness of the solution $\bar{\bar{\xi}}(x)$] is not general; it is linked to the small values of $\mathfrak{R}_j$, which is the only case investigated here.

19. Condensation of the Solutions to $U'_{1_x} = 0$ about an Asymptotic Solution

In accordance with the general results of dimensional analysis (Ref.2) applied to the boundary layers, the laws of evolution of the quantities in question depend exclusively on the corresponding Reynolds parameters.

For downstream segments far removed from both the leading edge $(x > 0)$ of the plate and from the segment x_j of connectivity between the laminar and "turbulent" states, the local quantities that characterize the boundary layer should tend to a state depending exclusively on $\mathfrak{R}_x$ since the episodic events occurring far upstream no longer have an influence. This, which agrees with the experimental facts, is a necessary consequence of dimensional analysis as soon as the existence of definite laws governing these phenomena is admitted.

The solution formed by a boundary layer which is "turbulent" from the origin $x = 0$ thus will be an asymptotic solution toward which all solutions with a transitorily laminar history will tend (this will be especially true for the friction coefficients).

In addition, since $\dfrac{U_1}{U_0}$, starting from a certain segment x and for an arbitrary particular solution, tends to a definite limit $\dfrac{X}{U_0}$, this limit $\dfrac{X}{U_0}$ should be a universal absolute constant irrespective of the particular case under consideration*.

19.1 Determination of the Value of $\dfrac{X}{U_0}$

It is not easy to fix the numerical value of $\dfrac{X}{U_0}$; however, since this value is universal, we are entitled to make use of the case of small $\mathfrak{R}_j$ treated

* The rigorousness of this conclusion is obviously limited by the approximations of the theory.

in the preceding Section. It is known that this value must be lower than $\frac{U_{1j}}{U_0}$ with respect to the boundary of the sublayer in the last laminar segment x_j (whence $\frac{U_{1j}}{U_0} \cong 0.66$). On the other hand, in the expression of $\bar{\xi}^2$ given in the preceding Section (which is valid only for small values of $\Re_x$), the term in x is that corresponding to the asymptotic solution:

$$\bar{\xi} = \sqrt{\frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}} \cdot \frac{\nu x}{U_0}},$$

i.e., /58

$$\mathcal{R}_{\bar{\xi}} = \sqrt{\frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}} \cdot \mathcal{R}_x}.$$

From this, we derive

$$(\mathcal{R}_{\bar{\xi}})'_{\mathcal{R}_x} = \frac{1}{2} \cdot \frac{\sqrt{\frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}}}}{\sqrt{\mathcal{R}_x}}$$

and, still for the asymptotic solution,

$$\bar{C}_f^* = 2\bar{\mathcal{A}} \cdot \xi'_x = 2\bar{\mathcal{A}} (\mathcal{R}_{\bar{\xi}})'_{\mathcal{R}_x} = \frac{\bar{\mathcal{A}}}{\sqrt{\mathcal{R}_x}} \cdot \sqrt{\frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}}}.$$

If we first assume that the statement at the end of the preceding Section is rigorously applicable, then the maximum of C_f^* must be identical with the value calculated here, meaning that $(\Re_{\xi})'_{\Re_x}$ (or ξ'_x) of the complete expression must be independent of x'_j. This condition is written as follows:

$$\sigma^2\, \overline{5.5}^2 \left(\frac{1 - \frac{X}{U_0}}{1 - \frac{U_{1j}}{U_0}} \right)^2 - \frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}} = 0,$$

at least in first approximation.

This condition, in principle, defines $\frac{X}{U_0}$ as soon as $\frac{U_{1j}}{U_0}$ and $\frac{B}{A}$ are fixed; however, $\frac{B}{A}$ is not accurately known numerically (see end of Sect.6.2).

On the other hand, in discussing the integration by parts we showed that the local friction coefficient, starting from the laminar value in x_j, should increase, to reach its maximum $\bar{C}_f^*$ in $\bar{x}_j \cong x_j$. This coefficient, at the same Reynolds number $\Re_{x_j}$, will be

$$\bar{C}_f^* \geqslant C_{fL}^* \quad \text{with} \quad C_{fL}^* = \frac{0.665}{\sqrt{\mathcal{R}_x}}.$$

From this, we derive

$$\bar{\mathcal{A}}\sqrt{\frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}}} \geqslant 0.665.$$

Eliminating $\frac{B}{A}$ between these conditions, we obtain /59

$$\sigma^2 \cdot \overline{5.5}^2 \left(\frac{1-\frac{X}{U_0}}{1-\frac{U_{1j}}{U_0}} \right)^2 - \left(\frac{0.665}{\bar{\mathcal{A}}}\right)^2 \geqslant 0$$

or

$$\left(1-\frac{X}{U_0}\right) \cdot \bar{\mathcal{A}} \geqslant 0.065,$$

with $\bar{\mathcal{A}}$ being a function of $\frac{X}{U_0}$. Thus, without making use of the more or less inaccurate determination of $\frac{B}{A}$, we find $\frac{X}{U_0} \leq 0.44$ to 0.45*, which we will retain as the most probable value $\left(\text{then, we find } \frac{B}{A} \cong 1.58, \mathcal{A} = 0.113\right)$.

Let us express the coefficient of total friction $C_f = \frac{q_\xi + q_\varepsilon}{\frac{\rho}{2} U_0^2 x}$ where $q_\xi +$ $+ q_\varepsilon$ is the loss of rate of flow of the momentum in the segment x under consideration. This coefficient is connected with the local coefficient, as

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* Nikuradse gives 0.38 as result of his experiments. Considering the configuration of the field U(y), it seems that $\frac{X}{U_0}$ should be slightly higher than 0.45, possibly because of the fact that one should have taken $\frac{U_{1j}}{U_0} > 0.66$.

follows:

$$C_f(x) = \frac{1}{x}\int_0^x C_f^*(x)\,dx = \frac{x_j}{x}C_f(x_j) + \frac{1}{x}\int_{x_j}^{\bar{x}_j} C_f^*\,dx + \frac{1}{x}\int_{\bar{x}_j}^{x} C_f^*\,dx$$

or

$$C_f(x) \cong \frac{x_j}{x}C_{f_L}(x_j) + \frac{1}{x}\int_{x_j}^{x} \overline{C}_f^*\,dx$$

which, for large x, tends to $\frac{1}{x}\int_0^x \overline{C}_f^* dx$, which is the asymptotic solution $\overline{C}_f(x)$.

20. Evolution of the Asymptotic Total Friction Coefficient

Let us return now to the equation in ξ (derived from the fundamental relation for rotation at the base of the actual boundary layer). We will take from this equation only the terms in U'_{1x} (and U'''_{1x^3}) since the study to be made concerns the asymptotic branch where $U_1 = X = \text{const}$. However, we will leave the term in ξ'''_{x^3} untouched, which is derived from the presence, in eq.(I), of residual velocity terms $\Delta u(\xi)$ at the level ξ since, for large evolutions of x, its characteristic evolution may play an important role (see statements in Sect.6). This is written in the form

$$\xi\,\xi'_x - \frac{1}{2\pi^2}\Sigma\frac{1}{n^2}\xi^3\,\xi'''_{x^3} = \frac{\nu}{2U_0}\frac{\pi^2\frac{B}{A}}{\frac{X}{U_0}}.$$

For the Reynolds numbers, this equation is expressed by the change of variables: /60

$$\mathcal{R}_\xi = \frac{\nu}{U_0}\xi, \qquad \mathcal{R}_x = \frac{U_0}{\nu}x,$$

whence

$$(x)'_{\mathcal{R}_x} = \frac{1}{U_0\nu}, \qquad (\mathcal{R}_\xi)'_{\mathcal{R}_x} = (\mathcal{R}_\xi)'_x \cdot x'_{\mathcal{R}_x} = \frac{U_0}{\nu}\xi'_x \cdot \frac{1}{U_0\nu} = \xi'_x.$$

Similarly,

$$(\mathcal{R}_\xi)^{(n)}_{\mathcal{R}_x^n} = \frac{1}{\left(\frac{U_0}{\nu}\right)^{n-1}}\xi^{(n)}_{x^n},$$

so that

$$\frac{U_0}{\nu}\xi\,\xi'_x = \mathcal{R}_\xi\,(\mathcal{R}_\xi)'_{\mathcal{R}_x}, \qquad \frac{U_0}{\nu}\xi^3\cdot\xi'''_{x^3} = \mathcal{R}_\xi^3\,(\mathcal{R}_\xi)'''_{\mathcal{R}_x^3}.$$

Putting

$$b = \frac{\nu}{2U_0} \cdot \frac{\frac{B}{A}\pi^2}{\frac{X}{U_0}} \cdot \frac{U_0}{\nu}, \qquad a = \frac{\Sigma \frac{1}{n^2}}{2\pi^2},$$

the equation is written in the form

$$\mathcal{R}_\xi (\mathcal{R}_\xi)'_{\mathcal{R}_x} - a \mathcal{R}_\xi^3 (\mathcal{R}_\xi)'''_{\mathcal{R}_x^3} = b.$$

To facilitate the writing, we will use the notations

$$\mathcal{R}_\xi = r, \qquad \mathcal{R}_x = x,$$

whence

$$r r'_x - a r^3 r'''_{x^3} = b.$$

The hypotheses of small $\xi'_x \xi'''_{x^3}$, used in establishing the relation in $\xi(x)$, are expressed by identical hypotheses concerning r'_x, r'''_{x^3}.

Let $Y = r^4$, from which we obtain

$$Y'_x = 4 r^3 r'_x, \qquad Y'''_{x^3} \cong 4 r^3 r'''_{x^3}.$$

In Y, the equation becomes

$$\frac{1}{4} \cdot \frac{Y'_x}{\sqrt{Y}} - \frac{a}{4} Y'''_{x^3} = b \qquad \text{or} \qquad \frac{1}{2} \cdot \frac{Y'_x}{\sqrt{Y}} - \frac{a}{2} Y'''_{x^3} = 2b,$$

i.e., on integrating a first time (under the condition that b is assumed as constant), we obtain

$$\sqrt{Y} - \frac{a}{2} Y''_{x^2} = 2bx + \text{const } \mathcal{C}.$$

20.1 Integration of the Equation in Y /61

Let us put Y in the form of an algebraic expansion in x, of the fourth degree:

$$Y = \alpha x^4 + \beta x^3 + \gamma x^2 + \eta x + \zeta.$$

$$Y'_x = 4\alpha x^3 + 3\beta x^2 + 2\gamma x + \eta.$$

$$Y''_{x^2} = 12\alpha x^2 + 6\beta x + 2\gamma.$$

For identification, we obtain

$$\sqrt{\alpha x^4 + \beta x^3 + \gamma x^2 + \eta x + \zeta} = [12 \alpha x^2 + 6 \beta x + 2 \gamma] + 2 b x + \mathcal{C}$$
$$= 6 \alpha a x^2 + (3 \beta a + 2 b) x + \gamma a + \mathcal{C},$$

i.e.,

$$\alpha x^4 + \beta x^3 + \gamma x^2 + \eta x + \zeta = 36 \alpha^2 a^2 x^4 + 2 (3 \beta a + 2 b) 6 \alpha a x^3$$
$$+ \{ 12 \alpha a (\gamma a + \mathcal{C}) + (3 \beta a + 2 b)^2 \} x^2$$
$$+ 2 \{ (\gamma a + \mathcal{C}) (3 \beta a + 2 b) x \} + (\gamma a + \mathcal{C})^2.$$

From this a first particular solution is obtained by putting

$$\begin{array}{lll}
\alpha = 36 \alpha^2 a^2 & \text{or} & \alpha = \dfrac{1}{36 a^2}, \\
\beta = 2 (3 \beta a + 2 b) 6 \alpha a & \text{or} & \beta (1 - 36 a^2 \alpha) = 24 b \alpha a, \\
\gamma = 12 \alpha a (\gamma a + \mathcal{C}) + (3 \beta a + 2 b)^2 & \text{or} & \gamma (1 - 12 \alpha a^2) = 12 \alpha a \mathcal{C} \\
 & & \quad + (3 \beta a + 2 b)^2, \\
\eta = 2 \{ (\gamma a + \mathcal{C}) (3 \beta a + 2 b) \xi \} & \text{or} & \eta = 2 \{ (\gamma a + \mathcal{C}) (3 \beta a + 2 b) \}, \\
\zeta = (\gamma a + \mathcal{C})^2, & &
\end{array}$$

whence

$$Y_I = \alpha x^4 + \beta x^3 + \gamma x^2 + \eta x + \zeta.$$

In addition, a second particular solution exists which is obtained by setting $\alpha' = 0$ whence $\beta' = 0$. This leaves $Y''_{x^2} = 6\beta' x + 2\gamma' = 2\gamma'$ constant.

The equation

$$\sqrt{Y} - \frac{a}{2} Y''_{x^2} = 2 b x + \mathcal{C}$$

reduces to

$$\sqrt{Y} = 2 b x + \mathcal{C} + a \gamma'.$$

Identification yields /62

$$\gamma' = 4 b^2, \qquad \eta' = 4 \mathcal{C} b, \qquad \zeta' = \mathcal{C}^2,$$

and the second particular solution will be

$$Y_{II} = 4 b^2 x^2 + 4 b (\mathcal{C} + a \gamma') x + (\mathcal{C} + a \gamma')^2.$$

The general solution of the suggested equation thus becomes

$$Y = Y_I k_I + Y_{II} k_{II},$$

where k_I, k_{II} are constants. This yields

$$Y = k_I \alpha x^4 + k_I \beta x^3 + (\gamma k_I + 4 b^2 k_{II}) x^2 + [\eta k_I + 4 b (\mathcal{C} + a \gamma') k_{II}] x + [\zeta k_I + (\mathcal{C} + a \gamma')^2 k_{II}],$$

whence

$$\mathcal{R}_\xi = \sqrt[4]{Y(x)}.$$

The total friction coefficient, according to its definition, is expressed as follows:

$$\overline{C}_f \cong \frac{\overline{\xi}}{x} 2 \overline{\mathcal{A}} = \frac{\overline{\mathcal{R}}_\xi}{\mathcal{R}_x} 2 \overline{\mathcal{A}}.$$

Consequently,

$$\overline{C}_f = 2 \overline{\mathcal{A}} \sqrt[4]{k_I \alpha + \frac{k_I \beta}{\mathcal{R}_x} + \frac{\gamma k_I + 4 b^2 k_{II}}{\mathcal{R}_x^2} + \frac{\eta k_I + 4 b k_{II} (\mathcal{C} + a \gamma')}{\mathcal{R}_x^3} + \frac{\{k_{II} (\mathcal{C} + a \gamma')^2 + k_I \zeta\}}{\mathcal{R}_x^4}},$$

(where $\bar{\mathcal{A}}$ is a constant)* which is finally written in the form

$$\overline{C}_f = \sqrt[4]{\overline{\alpha} + \frac{\overline{\beta}}{\mathcal{R}_x} + \frac{\overline{\gamma}}{\mathcal{R}_x^2} + \frac{\overline{\eta}}{\mathcal{R}_x^3} + \frac{\overline{\zeta}}{\mathcal{R}_x^4}}.$$

20.2 Relative Significance of the Coefficients and Evolution of Friction

Let us now investigate in somewhat more detail the identification conditions determining the constants α, β, γ, η, ζ of the first particular solution. These yield

$$\alpha = \frac{1}{36 a^2},$$

$$(1 - 36 a^2 \alpha) \beta = 24 b \alpha a,$$

i.e., /63

$$(1 - 1) \beta = \frac{b}{a} 0.667,$$

meaning that again $\beta \to \infty$.

Obviously, this is not rigorously so since the equation of the third order in ξ, retained here, is only approximate (for rotations at the base of the

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* Here, $\bar{\mathcal{A}}$ depends only on $\frac{X}{U_0}$ as is obvious when referring to its expression (Chapt.III, Sect.11); $\bar{\mathcal{A}} \cong 0.113$ for $\frac{X}{U_0} = 0.45$.

actual boundary layer, specifically the terms in $\frac{\nu}{U_0}\xi''_{x^2}$ have been eliminated). Nevertheless, this means that β assumes very large values. Then,

$$\gamma(1-12\alpha a^2) = 12\alpha a \mathcal{C} + (3\beta a + 2b)^2 \cong 9a^2\beta^2,$$

$$\gamma \cong \frac{9}{0.667}a^2\beta^2 = 13.5\cdot a^2\beta^2.$$

$$\eta = 2\{(\gamma a + \mathcal{C})(3\beta a + 2b)\} \cong \frac{2\cdot 27}{0.66}a^4\beta^3 = 81\cdot a^4\beta^3,$$

$$\zeta = (\gamma a + \mathcal{C})^2 \cong 182\cdot a^6\beta^4 \quad (\text{finally}, \gamma' = 4b^2).$$

The boundary conditions that must determine the three constants of integration $\mathcal{C}$, k_I, k_{II} of the equation in ξ are, in principle, the conditions to be written at the starting point of the boundary layer, i.e., for extremely small $\Re_x$ (here, the asymptotic solution is involved).

These conditions, in particular, lead to a definition of the constant term of $Y(\varkappa)$, namely

$$\zeta k_I + (\mathcal{C} + a\gamma')^2\cdot k_{II},$$

such that

$$\zeta k_I = k_I\cdot 182.5\cdot a^6\beta^4$$

will assume a finite value, with β being extremely large and k_I being extremely small; in Y, for moderate $\varkappa$, this will leave

$$Y \cong 4b^2 k_{II}\varkappa^2 + \{4b(\mathcal{C} + a\gamma')k_{II}\}\varkappa + \{\zeta k_I + (\mathcal{C} + a\gamma')^2 k_{II}\}.$$

Thus, with only an alteration of the constant term, Y is identical with the second particular solution to which there corresponds the equation in ξ, of the following reduced form:

$$\xi\xi'_x = b.$$

Thus, for moderate $\Re_x$, we will obtain

$$\overline{C}_f \cong \sqrt[4]{\frac{\overline{\gamma}}{\Re_x^2} + \frac{\overline{\eta}}{\Re_x^3} + \frac{\overline{\zeta}}{\Re_x^4}},$$

where $\overline{\gamma}$, $\overline{\eta}$, $\overline{\zeta}$ reduce to the terms in k_{II} (which $\overline{\alpha}$ and $\overline{\beta}$ do not contain).

When $\Re_x$ is much larger, the neglected terms will gradually gain significance with respect to the terms retained above, which incorporate $\Re_x$ in denominators at the highest powers. For $\Re \to \infty$, the principal term will basically remain as follows: /64

$$\overline{C}_{f\infty} = \sqrt[4]{\overline{\alpha}}$$

(a nonzero limiting friction exists here), with the term $\overline{\alpha}$ being necessarily extremely small in comparison with $\overline{\beta}$, as indicated above. In fact, we are unable to define the conditions of origin of the turbulent layer corresponding to the asymptotic solution, since the characteristics of the slope to the origin $\xi'_x(x = 0)$ and of the radius of curvature $\xi''_{x^2}(x = 0)$ are lacking.

At most, we could make the following statement: Previously, we mentioned that, for $x = 0$, the "turbulent" layer - if it were able to exist at all - would reduce to the sublayer since the velocity at the boundary is U_0. For the order of magnitude of the thickness of a line of particles in rolling motion, we found $y \cong \frac{\nu\, 20}{U_0}$ (see Sect.9). Thus, the local friction coefficient in $x \cong 0$ would be

$$C_f^* \, (x \cong 0) = \frac{\rho\, \nu}{\frac{\rho}{2} U_0^2} \cdot \frac{U_0^2}{20\, \nu} = 0.1.$$

Consequently, if the layer could start from the "turbulent" state, the local friction coefficient would be lower than that of the laminar state $\left[C^*_{f\infty} = \left(\frac{0.665}{\sqrt{\Re}}\right) \to \infty \text{ as } \Re \to 0\right]$. The laminar state would be at maximum entropy, which would explain its necessary establishment in the start-up zone $x \cong 0$.

For a quantity x which is larger but still sufficiently small to render the integration in parts of the preceding Section (Sect.19) valid, the local (asymptotic) friction would be

$$\overline{C}_f^* = \frac{\overline{\mathcal{A}}}{\sqrt{\mathcal{R}_x}} \cdot \sqrt{\frac{\pi^2 \frac{B}{A}}{\frac{X}{U_0}}} \cong \frac{0.67}{\sqrt{\mathcal{R}_x}}$$

for $\frac{X}{U_0} = 0.45$ and $\frac{B}{A} = 1.57$, $\overline{\mathcal{A}} = 0.113$.

This represents the domain of increase in laminar and "turbulent" friction. From this follows the "turbulent" total friction $\overline{C}_f \cong \frac{1.34}{\sqrt{\Re_x}}$. This expression, valid for small values of $\Re_x$, must be in continuous connection with the general expression

$$\overline{C}_f = \sqrt[4]{\overline{\alpha} + \frac{\overline{\beta}}{\mathcal{R}_x} + \frac{\overline{\gamma}}{\mathcal{R}_x^2} + \frac{\overline{\eta}}{\mathcal{R}_x^3} + \frac{\overline{\zeta}}{\mathcal{R}_x^4}}.$$

This means that, in this domain, $\bar{\alpha}$ and $\frac{\bar{\beta}}{\Re_x}$ are very small with respect to $\frac{\bar{\gamma}}{\Re_x^2}$ and that $\bar{\eta}$ and $\bar{\zeta}$ can be neglected. Hence, we obtain the following value for $\sqrt[4]{\bar{\gamma}}$: /65

$$\sqrt[4]{\bar{\gamma}} \cong 1.34.$$

No further information can be gained from the integration by parts; one would have to be able to extend its domain of validity to that in which ξ'_x, ξ''_{x^2} would cease being small. However, this is not the case here.

Two further conditions are required for determining $\bar{\alpha}$ and $\bar{\beta}$.

20.3 Numerical Results

Obviously, these results can be obtained from a comparison with practical data such as those obtained by Wieselsberger, Gebers, and Kempf which are classical.

We find

$$\bar{\alpha} \cong 8 \cdot 10^{-12}, \qquad \bar{\beta} \cong 700 \cdot 10^{-6} = 0.7 \cdot 10^{-3}$$

so that

$$\bar{C}_f = \sqrt[4]{8 \cdot 10^{-12} + \frac{700 \cdot 10^{-6}}{\mathcal{R}} + \frac{3.25}{\mathcal{R}^2} \cdots}$$

The accompanying table gives the elements of the curve $\bar{C}_f(\Re)$, while the Diagrams IV in Section 40.5 indicate that the law derived in this manner from the theoretical expression of $\bar{C}_f$ agrees satisfactorily with the experimental points.

We will also calculate the local friction, since

$$\bar{C}_f = \frac{1}{\mathcal{R}} \int_0^{\mathcal{R}} \bar{C}_f^* \, d\mathcal{R} \qquad \text{and} \qquad \bar{C}_f^* = (\mathcal{R}\,\bar{C}_f)'_{\mathcal{R}} = \frac{\bar{\alpha} + \frac{3}{4}\frac{\bar{\beta}}{\mathcal{R}} + \frac{1}{2}\frac{\bar{\gamma}}{\mathcal{R}^2} \cdots}{\left(\bar{\alpha} + \frac{\bar{\beta}}{\mathcal{R}} + \frac{\bar{\gamma}}{\mathcal{R}^2} \cdots\right)^{3/4}}$$

$\mathcal{R}$	$0.31^5 \cdot 10^6$	$1 \cdot 10^6$	$2 \cdot 10^6$	$3 \cdot 10^6$	$10 \cdot 10^6$	$100 \cdot 10^6$	$315 \cdot 10^6$
$\log_{10} \mathcal{R}$ ...	5.5	6	6.30	6.48	7	8	8.5
$\bar{C}_f$	$6.8 \cdot 10^{-3}$	$5.1^5 \cdot 10^{-3}$	$4.3^5 \cdot 10^{-3}$	$3.9^5 \cdot 10^{-3}$	$2.9^7 \cdot 10^{-3}$	$1.9^7 \cdot 10^{-3}$	$1.7^8 \cdot 10^{-3}$
$\log 10^3 \cdot \bar{C}_f$.	0.83^5	0.71	0.64	0.59^5	0.47	0.29^5	0.25
$\bar{C}_f^*$	$5.3 \cdot 10^{-3}$	$3.8^9 \cdot 10^{-3}$	$3.3^3 \cdot 10^{-3}$	$3 \cdot 10^{-3}$	$2.3^2 \cdot 10^{-3}$	$1.7^2 \cdot 10^{-3}$	$1.7^2 \cdot 10^{-3}$
$\log 10^3 \cdot \bar{C}_f^*$.	0.72^5	0.58^5	0.52	0.47^5	0.36^5	0.240	0.23^5

It will be noted that the minimum friction for $\Re \to \infty$ is $\overline{C}_f = \sqrt[4]{\overline{\alpha}}$ or $\overline{C}_f \cong$ .166 $\cong 1.7 \times 10^{-3}$. The table also shows that $\overline{\alpha}$ and $\frac{\overline{\beta}}{\Re}$ become negligible with respect to $\frac{\overline{\gamma}}{\Re^2}$ for $\Re < 1000$. This accurately defines the validity domain of the previously used integration by parts.

21. Determination of a Theoretical Value of $\overline{\beta}$

Even if one cannot proceed to a theoretical determination of the coefficient $\overline{\beta}$, an attempt must be made to define at least its order of magnitude by theoretical considerations.

First of all, let us review the state of the art as of today.

21.1 Status of the Problem

In the preceding Sections, we performed calculation of the evolution of the boundary layer on passing from the laminar to the stationary turbulent state, for the case in which this transition takes place at low Reynolds numbers corresponding to the domain in which $\overline{C}_f^* \simeq C_{f_L}^*$. To this corresponds a mechanism of initiation of rotation of the sublayer elements such that ζ (see Sect.12.3) is close to zero, and a critical velocity X of the sublayer such that $\frac{X}{U_0} = 0.45$.

This domain of transition (denoted by the subscript j), considering only the stationary Navier-Stokes equations, is extremely short ($\Delta\Re \sim 6$); this prevents us, with the approximations used, to extend our calculation to the more important $\Re_j$.

We also were able to calculate the form of theoretical evolution of the local friction coefficient $\overline{C}_f^*$ corresponding to the domain in which $U_{1_x}' \simeq 0$, i.e., $U_1 \cong X$ (where X might possibly decrease slowly with increasing x).

In the theoretical law

$$\overline{C_f} = \frac{\overline{\alpha} + 0.75\frac{\overline{\beta}}{\Re} + \frac{1}{2}\frac{\overline{\gamma}}{\Re^2}\cdots}{\left[\overline{\alpha} + \frac{\overline{\beta}}{\Re} + \frac{\overline{\gamma}}{\Re^2}\cdots\right]^{3/4}},$$

$\overline{\gamma}$ is known while $\overline{\alpha}$ and $\overline{\beta}$ are not known despite the fact that, according to the above reasoning, it is known that $\overline{\alpha} = k_1\alpha$ is certainly minimal in comparison with $\overline{\beta}$, such that $\overline{\alpha}$ will play a role relative to $\frac{\overline{\beta}}{\Re}$ only for really large $\Re$.

On the other hand, $\frac{\overline{\gamma}}{\Re^2}$ will be notable only for very low values of $\Re$.

Finally, if we know the law $\zeta(U_1)$ in the zone in which U_1 is in evolution, then the law of thickness ϵ of the sublayer (see Sect.15) will result from /67

$$\varepsilon = \varepsilon_j\left[1 + \int_{U_{1j}}^{U_1}(1 + \zeta(U_1))\frac{dU_1}{U_1}\right] \qquad \left(\varepsilon_j = (1-\sigma)\, 5.5\sqrt{\frac{\nu x_j}{U_0}}\right).$$

From this, it is easy to derive a condition for the local friction:

$$C_f^* = \frac{2\nu}{U_0}\cdot\frac{U_1}{U_0}\frac{1}{\varepsilon}.$$

With $U_{1_j} \cong 0.66\, U_0$, and thus $\epsilon_j \cong 0.36$, the expression $\delta_L = 2\frac{\nu}{U_0}\sqrt{\Re_j}$, C_f^* is written as follows:

$$C_f^* = \frac{2\nu}{U_0\,\varepsilon_j}\cdot\frac{\frac{U_1}{U_{1j}}}{\left[1 + \int_{U_{1j}}^{U_1}(1+\zeta)\frac{dU_1}{U_1}\right]},$$

valid for the same zone.

If we consider a moderate value of ζ, this quantity will be removed from under the integral sign. For U_1, reaching the asymptotic value X, we will have

$$\overline{C_{fj}^*} = \frac{2\nu}{U_0\,\varepsilon_j}\cdot\frac{\frac{X}{U_{1j}}}{1 + (1+\zeta)\,\mathrm{Log}\frac{X}{U_{1j}}}.$$

As for the laminar state

$$C^*{}_{fLj} = \frac{2\nu}{U_0}\cdot\frac{U_{1j}}{U_0}\cdot\frac{1}{\varepsilon_j} = \frac{0.667}{\sqrt{\mathcal{R}_j}},$$

it then follows that

$$\left(\frac{C_f^*}{C^*{}_{fLj}}\right)_j = \frac{\frac{X}{U_{1j}}}{1 + (1+\zeta)\,\mathrm{Log}\frac{X}{U_{1j}}} \qquad \left(\text{with } \overline{C_{fj}^*} = \frac{1}{\sqrt{\mathcal{R}_j}}\cdot\frac{\frac{X}{U_{1j}}}{1 + (1+\zeta)\,\mathrm{Log}\frac{X}{U_{1j}}}\right).$$

21.2 Possible Forms of Evolution of Local Friction

Next, let us establish two working hypotheses:

1) $\bar{\alpha} = k_1\alpha$ will be considered as equal to zero.

It should be mentioned that this is not incompatible with the previous theoretical data and also not with the physical aspect of the development of friction.

2) Our second hypothesis provisionally will be as follows: As soon as $\mathfrak{R}_j$, /68
which represents the characteristic of the point of initiation of the "turbulent" state, is no longer small, the velocity gradient U'_Y in the sublayer will develop rapidly and this initiation will manifest itself in the sublayer by inducing rotation of its elements in accordance with an average mechanism to which a certain value of ζ is attached.

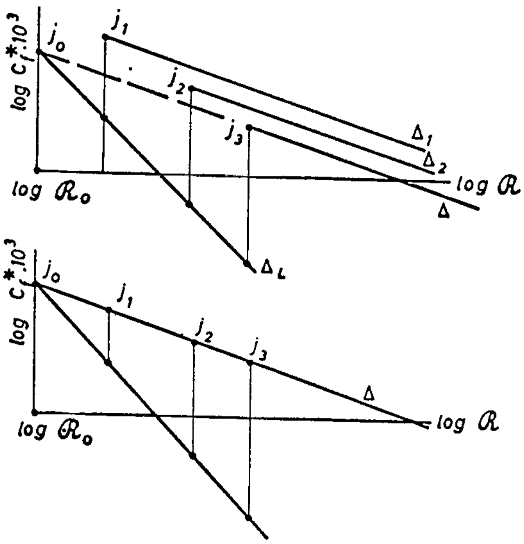

Fig.12

It should be noted that, when $\mathfrak{R}_j$ is small, these gradients differ little from that existing in the laminar layer, there is only a minor change in the state of rotation, and one approaches the constant-pressure process (second scheme, see Sect.12). This means that, for $\mathfrak{R} = \mathfrak{R}_0$ which is relatively weak but nevertheless the largest value to which the simplified step-by-step solution investigated in the preceding Section is applicable, we have $\overline{C}^*_f \simeq C^*_{f_L}$. (For the time being, we are unable to define the order of magnitude of $\mathfrak{R}_0$ except by experimental means.)

Nevertheless, let us consider a start-up value $\mathfrak{R}_{j_1}$, large with respect to $\mathfrak{R}_0$. To this we will assign a certain average value of ζ, namely ζ_1. Just about when this value is fixed, the relation

$$\frac{\overline{C}^*_{fj_1}}{C^*_{fLj_1}} = \frac{\dfrac{X}{U_{1j}}}{1 + (1+\zeta)\,\mathrm{Log}\,\dfrac{X}{U_{1j}}}$$

(see above, Sect.12.3) will show that $\overline{C}^*_{f_{j_1}}$ is represented, in the logarithmic diagram (Fig.12) by

$$\log \overline{C}^*_{fj_1}\cdot 10^3 = \log C^*_{fLj}\cdot 10^3 + \log \frac{\dfrac{X}{U_{1j}}}{1 + (1+\zeta_1)\,\mathrm{Log}\,\dfrac{X}{U_{1j}}}.$$

This figurative point will thus be located above $\log \cdot C^*_{f_{L_j}} \cdot 10^3$. From this, /69
at increasing $\mathfrak{R}$, the point will shift along the straight line Δ_1 of a slope

$-\frac{1}{4}$, since

$$\overline{C_f^*} \cong \frac{0.75\,\frac{\bar\beta}{\mathcal{R}}}{\left(\frac{\bar\beta}{\mathcal{R}}\right)^{3/4}} = 0.75\cdot\sqrt[4]{\frac{\bar\beta}{\mathcal{R}}} = \overline{C^*}_{l_j}\sqrt[4]{\frac{\mathcal{R}_j}{\mathcal{R}}},$$

whence

$$\bar\beta = \frac{\overline{C^*}_{l_j}}{0.75}\cdot\mathcal{R}_j^{1/4}.$$

In this image, $\bar\beta$ then is a parameter depending in principle on $\Re_j$, which is the characteristic of the point of initiation of the turbulent state.

The same reasoning can be used for all $\Re_{j_n} > \Re_0$, to which there corresponds a system of parallel straight lines of slope $-\frac{1}{4}$, originating at the points j_n. Among these, one line (namely, Δ) passes through the point j_0 of the coordinates $\log \Re_0$, $\log C^*_{f_L}(\Re_0)\cdot 10^3$. However, it has been shown that, for very large $\Re$, a condensation of the characteristics of the "turbulent" boundary layers and thus also of the local friction C^*_f into a unique asymptotic solution must exist there.

Consequently, it is necessary that all straight lines Δ_1, Δ_2, ... converge into a single line. This is impossible with the hypotheses made here, which led to the preceding diagram. However, it will become possible as soon as the curve $\log C^*_f$ coincides with the straight line Δ of slope $-\frac{1}{4}$, passing through j_0. In that case, the diagram will assume the second of the above-indicated forms.

21.3 Consequences

A preliminary and highly important conclusion can be drawn from the fact that the figurative straight line Δ of $\log C^*_f$ is unique: On initiation of the "turbulent" state, the local friction coefficient will osculate the figurative unique curve of the asymptotic local "turbulent" friction. We will show later (Diagram III in Sect.39.1) that this fundamental fact agrees well with practical experience.

Another conclusion concerns the development of ζ with $\Re$. In fact, as shown before (see Sect.12), we have

$$\frac{\overline{C_f^*}}{C^*_{l_t}} = \frac{\frac{X}{U_{1j}}}{1+(1+\zeta)\,\mathrm{Log}\frac{X}{U_{1j}}} \qquad \left(\frac{X}{U_{1j}} < 1,\quad \frac{X}{U_0}\cong 0.45,\quad \frac{U_{1j}}{U_0}\cong 0.66\right).$$

and thus

$$1+\zeta=\frac{1}{\operatorname{Log}\frac{U_{1j}}{X}}\left\{1-\frac{\frac{X}{U_{1j}}}{\frac{\overline{C_f^*}}{C^*_{/L}}}\right\}.$$

For $\mathfrak{R} \sim \mathfrak{R}_0$: /70

$$\frac{\overline{C_f^*}}{C^*_{/L}} \sim 1, \qquad 1+\zeta \to 0.82, \qquad \zeta \to -0.18.$$

For $\mathfrak{R} \to \infty$:

$$\frac{\overline{C_f^*}}{C^*_{/L}} \to \infty, \qquad (1+\zeta) \to \frac{1}{\operatorname{Log}\frac{U_{1j}}{X}} = 2.6 \quad \text{and} \quad \zeta \to 1.6.$$

Between these values of $\mathfrak{R}$, we have

$$\frac{\overline{C_f^*}}{C^*_{/L}} \cong \frac{0.75\sqrt[4]{\frac{\mathcal{R}_0}{\mathcal{R}}}}{0.667\sqrt{\frac{\mathcal{R}_0}{\mathcal{R}}}} = 1.12\sqrt[4]{\frac{\mathcal{R}}{\mathcal{R}_0}}.$$

This ratio will be known as soon as $\mathfrak{R}_0$ is known. A determination of $\overline{\beta}$ by comparison with the experimental results led to values of $\mathfrak{R}_0$ of the order of 10^3 (this result, obtained by a different process will be discussed later in the text). It is then possible to determine the development of ζ with $\mathfrak{R}$, which first increases rapidly close to $\zeta \sim 0$ and then slowly rejoins its asymptotic value of 1.6, in accordance with the accompanying table.

$\mathcal{R}$	10^3	10^5	10^5	10^7	10^8	∞
$1+\zeta$	+0.82	1.96	2.33	2.45	2.51	2.60
ζ	−0.18	0.96	1.33	1.45	1.51	1.60

21.4 Correction in the Scheme of the Domain of $\mathfrak{R}_0$

Actually, in the domain of $\mathfrak{R}_0$, the term in $\frac{\overline{\gamma}}{\mathfrak{R}^2}$ of the complete expression for the local turbulent friction

$$\overline{C_f^*} = \frac{0.5\frac{\overline{\gamma}}{\mathcal{R}^2} + 0.75\frac{\overline{\beta}}{\mathcal{R}}}{\left(\frac{\overline{\gamma}}{\mathcal{R}^2} + \frac{\overline{\beta}}{\mathcal{R}}\right)^{3/4}},$$

is preponderant, such that

$$\overline{C_f^*}(\mathcal{R}_0) \cong 0.5\sqrt[4]{\frac{\overline{\gamma}}{\mathcal{R}_0^2}} = \frac{0.667}{\sqrt{\mathcal{R}_0}}.$$

In the preceding scheme, it is necessary to substitute the straight line Δ by a straight line Δ' located slightly below Δ, such that its intersection with the figurative straight line Δ_L of the laminar friction will be slightly superior to $\Re_0$ in j_0' to which $\Re_0'$ corresponds (Fig.13). /71

For $\Re > \Re_0'$, the term $\overline{C}_f^*$ reduces to $\overline{C}_f^* \cong 0.75\sqrt[4]{\frac{\overline{\beta}}{\Re}}$, so that, in the logarithmic diagram, we obtain

$$\log \overline{C_f^*} = \log 0.75\sqrt[4]{\overline{\beta}} - \frac{1}{4}\log \mathcal{R}, \qquad \log C^*_{/L} = \log 0.667 - \frac{1}{2}\log \mathcal{R}.$$

Putting $\eta = \log \overline{C}_f^*$ and $x = \log \Re$, the equations of the straight lines Δ', Δ_L will become

$$\overline{\eta} = \overline{\eta}_0 - \frac{x}{4}, \qquad \eta_L = \eta_{0_L} - \frac{x}{2},$$

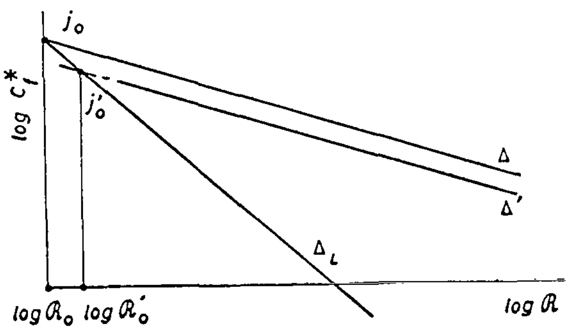

Fig.13

while their intersection in $x_0' = \log R_0'$ will be such that $4(\eta_0 - \eta_{0_L}) + x_0' = 0$, so that

$$\mathcal{R}'_0 = \left(\frac{0.667}{0.75}\right)^4 \cdot \frac{1}{\overline{\beta}} = \frac{0.63}{\overline{\beta}}.$$

If it is assumed that very little deviation exists for $\Re = \Re_0'$ between the approximate expression of $\overline{C}_f^*$, namely

$$\overline{C_{fJ}^*} = 0.75\sqrt[4]{\frac{\overline{\beta}}{\mathcal{R}'_0}}$$

and its complete expression

$$\overline{C_f^*} = \frac{0.5\frac{\overline{\gamma}}{\mathcal{R}'^2_0} + 0.75\frac{\overline{\beta}}{\mathcal{R}'_0}}{\left(\frac{\overline{\gamma}}{\mathcal{R}'^2_0} + \frac{\overline{\beta}}{\mathcal{R}'_0}\right)^{3/4}},$$

then we can put

$$\frac{0.5 \cdot \frac{\overline{\gamma}}{\mathcal{R}'^2_0} + 0.75 \frac{\overline{\beta}}{\mathcal{R}'_0}}{\left(\frac{\overline{\gamma}}{\mathcal{R}'^2_0} + \frac{\overline{\beta}}{\mathcal{R}'_0}\right)^{3/4}} = 0.75 \sqrt[4]{\frac{\overline{\beta}}{\mathcal{R}'_0}} \, (1 + \delta)$$

where δ is small.

Let us set /72

$$\frac{\overline{\gamma}}{\mathcal{R}'^2_0} = \mu \frac{\overline{\beta}}{\mathcal{R}'_0}.$$

From this, we obtain

$$\overline{\beta} = \frac{\overline{\gamma}}{\mathcal{R}'_0} \cdot \frac{\overline{1}}{\mu} = \frac{0.630}{\mathcal{R}'_0}$$

in accordance with the above-given expression.

Consequently, $\mu = \frac{\overline{\gamma}}{0.630} = 5$. We also obtain

$$\frac{0.5\,\mu + 0.75}{(1 + \mu)^{3/4}} = 0.75\,(1 + \delta),$$

i.e., $\delta = 0.14$ as well as $\frac{\overline{C}^*_f(R'_0)}{C^*_{f_L}(R'_0)} = 1.14$. From this, we derive $\zeta'_0 = 0.05$.

21.5 Determination of a Probable Value of $\mathfrak{R}'_0$ in Accordance with the Relative Importance of the Terms

Thus, for $\mathfrak{R} = \mathfrak{R}'_0$, the totality of the quantities in question is known except for $\mathfrak{R}'_0$ which is still lacking.

However, so as to keep the analysis coherent, it is necessary that the general hypotheses on which the analysis is based can be verified. One of these hypotheses refers to the order of magnitude of $\frac{V(\epsilon)}{U_1}$ which is the slope of the streamline. It has been demonstrated that this term should be negligible, i.e. (see Sect.2 and the corresponding statement), should be of the third infinitesimal order, with the first order being about at 0.1. Consequently, the third order is near 0.001 so that it becomes necessary that $\frac{V(\epsilon)}{U_1} < 0.001$. However, with $\epsilon = hU_1^{1+\zeta}$, such that

$$\epsilon_{j_L} = 0.365 \cdot 5.5 \sqrt{\frac{\nu x_j}{U_0}} = hU_{1j}^{1+\zeta}$$

we obtain

$$h = \frac{2\nu}{U_0} \cdot \frac{\sqrt{\mathcal{R}_j}}{U_{1j}^{1+\zeta}}.$$

The established turbulent state is characterized by

$$\bar{\varepsilon}_j = \varepsilon_{j_h} \left(\frac{X}{U_{1j}}\right)^{1+\zeta},$$

i.e.,

$$\bar{\varepsilon}_j = \left(\frac{X}{U_{1j}}\right)^{1+\zeta} \cdot \frac{2\nu}{U_0} \sqrt{\mathcal{R}_j}.$$

Since, according to the continuity equation, /73

$$V(\varepsilon) = -\left(\frac{U_1}{\varepsilon}\right)'_x \cdot \frac{\varepsilon^2}{2} \qquad \text{with} \qquad \left(\frac{U_1}{\varepsilon}\right)'_x = -\frac{\zeta U'_{1x}}{h U_1^{1+\zeta}} = -\frac{\zeta U'_{1x}}{\varepsilon},$$

it follows that $V(\varepsilon) = \frac{\zeta}{2} \cdot \varepsilon U'_{1x}$ and, consequently,

$$\frac{V(\varepsilon)}{U_1} = \frac{h}{2} \zeta U_1^{\zeta} \cdot U'_{1x} = \frac{\nu}{U_0} \cdot \frac{\sqrt{\mathcal{R}}}{U_{1j}^{1+\zeta}} \zeta U_1^{\zeta} U'_{1x}.$$

Let us now evaluate $U'_{1x} = \frac{\Delta U_1}{\Delta x}$. Here, $|\Delta U_1| \cong U_{1_J} - X$ and $\Delta x = \frac{\nu}{U_0} \Delta\mathfrak{R}$. It will be shown below (Sect.39) that actually the transition extends from $\frac{\mathfrak{R}_J}{2}$ to $\mathfrak{R}_J$, meaning that we have $\Delta\mathfrak{R} = \frac{\mathfrak{R}_J}{2}$ from which it follows that

$$U'_{1x} \cong -\frac{0{,}2\, U_0}{\frac{\nu}{U_0} \cdot \frac{\mathcal{R}_j}{2}} \qquad (\text{with } U_{1j} \cong 0.65\, U_0, \quad X = 0.45\, U_0).$$

Thus,

$$\frac{V(\varepsilon)}{U_1} = -\frac{\nu}{U_0} \cdot \frac{\sqrt{\mathcal{R}}}{U_{1j}} \zeta \left(\frac{X}{U_{1j}}\right)^{\zeta} \left(-0.4 \cdot \frac{U_0^2}{\nu \mathcal{R}}\right) = -0.4\,\zeta \cdot \frac{U_0}{U_{1j}} \left(\frac{X}{U_{1j}}\right)^{\zeta} \cdot \frac{1}{\sqrt{\mathcal{R}}}.$$

The condition $\frac{V(\varepsilon)}{U_1} \leq 0.001$, in $\mathfrak{R} = \mathfrak{R}'_0$, will become

$$\sqrt{\mathcal{R}'_0} \geqslant \left[\frac{0.4\,\zeta}{0.001} \cdot \frac{U_0}{U_{1j}} \left(\frac{X}{U_{1j}}\right)^{\zeta}\right] = 25 \text{ to } 30,$$

for $\zeta = 0.05 \left(\text{at } \frac{U_{1_j}}{U_0} = 0.66 \text{ to } 0.65\right)$, i.e., $\mathfrak{R}'_0 \geq 650 - 900$. The order of magnitude of 10^3, obtained already by experimental comparison, can be approximately defined; then the order of magnitude of the value furnished by such a comparison (0.7×10^{-3}) will necessarily be obtained for $\beta \simeq \frac{0.63}{\mathfrak{R}'_0}$.

Obviously, this is not a real proof but only a statement having the purpose of demonstrating that it is unnecessary to use experimental results for defining the order of magnitude of $\mathfrak{R}'_0$.

21.6 Case of the Constant Term $\overline{\alpha}$ being Nonzero

Later in the text (Diagram IV in Sect.40.3), we will give a plot of the straight line Δ' with respect to the experimental points obtained by Wieselsberger, Gebers, and Kempf (as well as the curve $\overline{C}_f$ where $\overline{\beta}$ and $\overline{\alpha}$ are of experimental origin).

Kempf's experimental points with respect to $\mathfrak{R}$, extending from 20 to 350 × /74
× 10^6, show an upward convexity of the curve $\log 10^3\ \overline{C}_f$ (and $\log 10^3\ \overline{C}_f^*$) which would not be compatible with the general theoretical expression $\overline{C}_f = \sqrt[4]{\overline{\alpha} + \frac{\overline{\beta}}{\mathfrak{R}}}$ except under the condition of assuming $\overline{\alpha}$ as being nonzero but very small with respect to $\overline{\beta}$, as stipulated by the theory ($\overline{\alpha} \cong 8 \times 10^{-12}$).

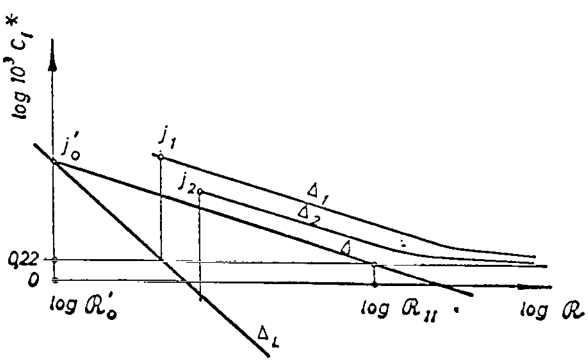

Fig.14

One of the basic premises of our above reasoning fails in this case, so that it is no longer possible to affirm the existence of a unique asymptotic solution except for $\mathfrak{R} \to \infty$, where all curves Δ_n admit of the horizontal asymptote $\log \sqrt[4]{\overline{\alpha}} \cdot 10^3$. A priori, it is now impossible to state that the curves Δ_n in the zone in which they are rectilinear $\left(\overline{\alpha} \text{ small with respect to } \frac{\beta}{\mathfrak{R}}\right)$ are not distinct. To each of these, a value $\overline{\beta}_n$ is then assigned (Fig.14).

Let us attempt to define the domain of variation of $\overline{\beta}_n$. For this, let us consider the domain of $\mathfrak{R}$ enclosed between $\mathfrak{R}_0$ and $\mathfrak{R}_{II}$, with $\mathfrak{R}_{II}$ being the value of $\mathfrak{R}$ relative to the intersection of the straight line Δ $\left(\text{of slope } -\frac{1}{4}\text{, originating from } j_0\right)$ with the asymptote $\log \sqrt[4]{\overline{\alpha}} \cdot 10^3 \cong 0.22$. Since the equation of Δ, with the notations used before, reads

$$\eta = \eta_0 - \frac{1}{4}(x - x_0),$$

where

$$x_0 = \log \mathcal{R}_0 = 2.81 \quad \text{and} \quad \eta_0 = \log C^*{}_{/L} \cdot 10^3 = \log \frac{0.667 \cdot 10^3}{\sqrt{\mathcal{R}_0}} = 1.42$$

this intersection will take place for

$$x = x_{II} = x_0 + 4(\eta_0 - \eta_{II}) \quad \text{where} \quad \eta_{II} = 0.22$$

whence

$$x_{II} = 7.6 \quad \text{and} \quad \mathcal{R}_{II} \cong 40 \cdot 10^6.$$

This would define the order of magnitude of large values of $\mathfrak{R}$ where the constant $\bar{\alpha}$ starts to intervene in a noticeable manner. /75

It is then found that:

1) In $\mathfrak{R} = \mathfrak{R}_0$, the quantity $\frac{\gamma}{\mathfrak{R}^2}$ is large compared to $\frac{\bar{\beta}_n}{\mathfrak{R}}$ from which, for $\mathfrak{R}_0 \cong 0.65 \times 10^3$, it follows that

$$\frac{\bar{\beta}_n}{\mathcal{R}_0} \leqslant \frac{1}{10} \cdot \frac{3.2}{\mathcal{R}^2} = \frac{3.2}{10 \cdot 0.42 \cdot 10^6} \quad \text{or} \quad \bar{\beta}_n \leqslant 1.18 \cdot 10^{-3}.$$

2) For $\mathfrak{R} \sim \mathfrak{R}_{II} = 40 \times 10^6$, the quantity $\frac{\bar{\beta}_n}{\mathfrak{R}}$ must be of the same order as $\bar{\alpha}$, i.e., for $\mathfrak{R} < \frac{\mathfrak{R}_{II}}{5} = 8 \times 10^6$, $\bar{\alpha}$ must be small with respect to $\frac{\bar{\beta}}{\mathfrak{R}}$. Hence,

$$\frac{\bar{\beta}_n}{8 \cdot 10^6} > 10 \cdot 8 \cdot 10^{-12},$$

i.e.,

$$\bar{\beta}_n > 640 \cdot 10^{-6} = 0.64 \cdot 10^{-3}.$$

The quantities $\bar{\beta}_n$ are enclosed between two approximate limits.

From this, for example, at $\mathfrak{R} = 1 \times 10^6$ which is the Reynolds number for which $\bar{\alpha}$ and $\frac{\bar{\gamma}}{\mathfrak{R}^2}$ can be neglected with respect to $\frac{\bar{\beta}}{\mathfrak{R}}$, it follows that $\bar{C}_f^* \cong$ $\cong 0.75 \sqrt[4]{\frac{\bar{\beta}}{\mathfrak{R}}}$ will be comprised between

$$0.75\sqrt[4]{0.64\cdot10^{-9}} = 3.75\cdot10^{-3} \quad \text{and} \quad 0.75\sqrt[4]{1.2\cdot10^{-9}} = 4.4\cdot10^{-3},$$

meaning that the quantities $\log C_f^* \cdot 10^3$ range between 0.58 and 0.64.

It is then no longer possible to affirm the existence of a true asymptotic solution Δ'; nevertheless, as shown in Diagram II in Section 21.7 a condensation of the solutions about Δ' takes place, which is a narrow condensation such that the above conclusions with $\overline{\alpha} \equiv 0$ will remain approximately valid even if $\overline{\alpha}$ assumes the value suggested by the exceptional measurements by Kempf.

However, still another explanation might exist: It is known from experimental results that the wall roughness increases the local friction to such an extent that, taking K as the dimension of the roughness, $\overline{C}_f^*(\mathfrak{R})$ assumes the form (Δ'') indicated in Fig.15 at $\frac{K}{x}$ = const.

However, Kempf's experiments were made with water, i.e., at a very low ν (1.0×10^{-6} at 20° as compared with 14.4×10^{-6} in air).

Referring to the curves given by Prandtl on the effect of roughnesses (Ref.1), a change in the function $\overline{C}_f^* = \sqrt[4]{\frac{\overline{\beta}}{\mathfrak{R}}}$ of the same order as that appearing in Kempf's measurements will be obtained for $\frac{K}{x} = 1 \times 10^{-6}$. /76

We do not know the length x of the plate nor the velocity used by Kempf. Nevertheless, to obtain $\mathfrak{R} = 300 \times 10^6$ at 20 m/sec (corresponding to a height of charge $\frac{\rho}{2} U_0^2 = \frac{100}{2} \times 400 = 20{,}000$ kg/m², or H = 20 m), it is necessary that

$$x = \frac{0.1\cdot10^{-6}}{20}\cdot300\cdot10^6 = 1.50\ \text{m}.$$

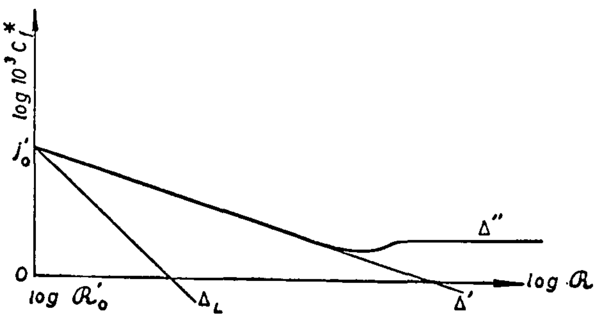

Fig.15

The roughness $\frac{K}{x} = 1 \times 10^{-6}$ was reached for $K = 1.5 \times 10^{-6}$, i.e., 1.5 μ. This roughness value corresponds to a quite noticeable smoothness of surface.

It might well be - and in our opinion with considerable probability - that the curvature of the curve $\log \overline{C}_f^* \cdot 10^3$, revealed in Kempf's measurements, is a manifestation of the roughness effect.

If this were so, $\overline{\alpha}$ would have to be considered as zero at the state of perfect smoothness, meaning that an asymptotic solution in the strict sense would exist.

Finally, another possibility to be studied might be that of the possible existence of nonstationary components in the boundary layer, at the loss of moment of momentum. In our analysis, we limited ourselves until now to the study of stationary components, while the development of friction had been deduced from eq.(I) (see Sects.6 and 17), written at the lower interface of the actual boundary layer with the sublayer.

Below, it will be shown that the nonstationary components which are zero along the wall, are very small along the border of the sublayer. So long as the latter is of minimal thickness, the effect of these components can be neglected. The effect may also disappear at very large Reynolds numbers where the sublayer thickens slightly and where, simultaneously, the components in question increase along its boundary.

21.7 Laminar Blasius Field; Determination of the Coefficients Φ_{nj} and of the Constants $\frac{B}{A}$ and $\bar{\mathcal{A}}$. /77

"Laminar" field:

$$\frac{U_{1j}}{U_0} = 0.65, \qquad \xi_j = 0.635 \cdot 5.5\sqrt{\frac{\nu x}{U_0}}, \qquad \sigma = 0.635$$

$$Y = \eta\sqrt{\frac{\nu x}{U_0}} = \left(0.365 + \frac{y}{\xi_j}\cdot\frac{\xi_j}{\delta}\right)5.5\sqrt{\frac{\nu x}{U_0}}$$

$$\eta = \left(0.365 + \frac{0}{\pi}0.635\right)5.5 = 2 + \frac{0}{\pi}3.5 \qquad \textit{(see Diagram I).}$$

$\frac{y}{\delta}$	0.365	0.470	0.522	0.575	0.681	0.787	0.840	0.895	1
$\frac{y}{\xi} = \frac{0}{\pi}$	0	1/6 = 0.167	1/4 = 0.25	1/3 = 0.333	1/2 = 0.5	2/3 = 0.667	3/4 = 0.750	5/6 = 0.835	1
η	2	2.583	2.875	3.165	3.750	4.335	4.625	4.925	5.5
$\frac{U}{U_0}$	0.649	0.778	0.832^5	0.875	0.937	0.975	0.988	0.991	1
$\frac{U_1}{U_0}$	0.650	0.650	0.650	0.650	0.650	0.650	0.650	0.650	0.650
$\frac{\bar{U}}{U_0}\frac{y}{\xi} = 0.35\frac{y}{\xi}$	0	0.058^6	0.087^5	0.116^5	0.175	0.233	0.262^5	0.293^5	0.350
$\frac{U}{U_0}$	0	0.070	0.094	0.108	0.112	0.084	0.074^5	0.046	0

Hence,

$$\frac{1}{U_0}\Phi_{1j} = 0.112, \qquad \frac{1}{U_0}\Phi_{2j} = 0.013^7, \qquad \frac{1}{U_0}\Phi_{3j} = 0.003^8, \qquad \frac{1}{U_0}\Phi_{6j} = -0.000^8$$

$$K_1 = 1, \qquad K_2 = 0.122, \qquad K_3 = 0.034, \qquad K_6 = -0.007$$

$$A = 0.900; \qquad B \cong 1.415; \qquad \frac{B}{A} \cong 1.57.$$

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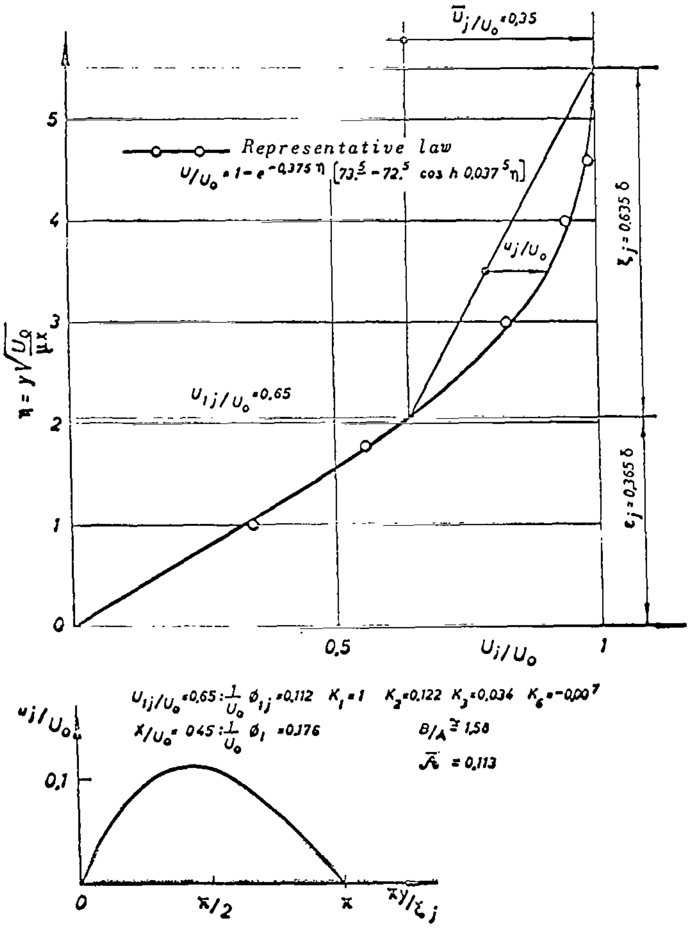

Fig.16 (Diagram I).

For $\dfrac{U_1}{U_0} = \dfrac{X}{U_0} = 0.45$; asymptotic "turbulent" field:

$\frac{y}{\xi}$	0	0.167	0.250	0.333	0.500	0.667	0.750	0.835	1
$\frac{\bar{U}}{U_0}$	0	0.110	0.148	0.170	0.176	0.133	0.110^5	0.083	0
$\frac{X}{U_0}$	~ 0.450	0.450	0.450	0.450	0.450	0.450	0.450	0.450	0.450
$\frac{U_0 - X}{U_0} \cdot \frac{y}{\xi}$.	0	0.092	0.138	0.184	0.275	0.367	0.413	0.460	0.550
$\frac{U}{U_0}$	~ 0.450	0.652	0.736	0.804	0.901	0.950	0.973^5	0.993	1

$$\frac{1}{U_0}\Phi_1 = 0.176, \quad K_1 = 1, \quad K_2 = 0.122, \quad K_3 = 0.034, \quad K_6 = -0.007, \quad \frac{B}{A} \cong 1.57$$ /79

$$\frac{2}{\pi}\Sigma\frac{1}{U_0}\cdot\frac{\Phi_{2n+1}}{2n+1} = 0.113, \quad \frac{2}{\pi}\Sigma(-1)^n\cdot\frac{1}{U_0}\cdot\frac{\Phi_n}{n} = -0.106, \qquad \Sigma\frac{1}{2U_0^2}\Phi_n^2 = 0.015^4$$

$$\mathcal{A} = 0.275 - 0.100 + 0.011^3 - 0.55\cdot 0.106 - 0.015^4 = 0.113$$

$$\left(1 - \frac{X}{U_0}\right)\mathcal{A} = 0.063 \sim \frac{0.665\cdot 0.35}{0.635\cdot 5.5}$$

$$\frac{\pi^2\frac{B}{A}}{\frac{X}{U_0}} \cong 35.$$

21.8 Diagram II; Local Friction log $\overline{C}_f^* \cdot 10^3$ as a Function of log $\mathfrak{R}$.

Comparison of the laws of theoretical and mixed origin:

$\mathcal{R}$	$0.9\cdot 10^3$	$0.31^5\cdot 10^6$	$1\cdot 10^6$	$3\cdot 10^6$	$10\cdot 10^6$	$100\cdot 10^6$	$315\cdot 10^6$
log $\mathcal{R}$	2.95	5.5	6	6.48	7	8	8.5
$10^3\cdot\overline{C}_f^*$ tanh	25	5.17	3.86	—	2.18	1.21^5	—
log $10^3\,\overline{C}_f^*$ tanh	1.40	0.71^5	0.58^5	—	0.34	0.09^5	—
$10^3\cdot C_{f_L}^*$......	$22.^2$	—	—	—	—	—	—
log $10^3\cdot C_{f_L}^*$ ..	1.34^5	—	—	—	—	—	—
$10^3\cdot\overline{C}_f^*\cdot$ exp .	—	5.3	3.88	3.00	2.30	1.72^5	1.72
log $10^3\cdot\overline{C}_f^*$ exp	—	0.72^5	0.58^5	0.47^5	0.36	0.240	0.23^5

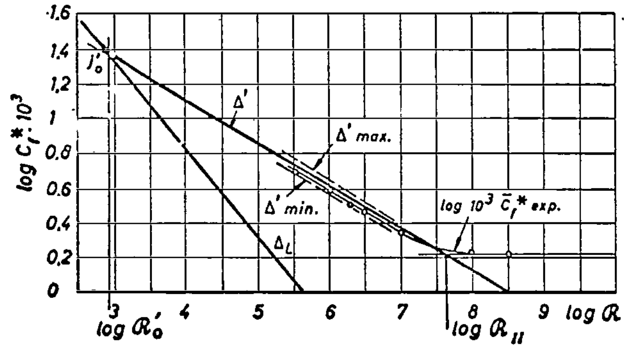

Fig.17 (Diagram II).

PART II

NONSTATIONARY STUDIES

22. General Remarks

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The preceding exposé furnishes the means, starting from an abscissa x_j where the laminar state ceases to exist (whose velocity distributions are known from the Blasius method), for calculating a stationary flow of the boundary layer with sublayer, satisfying hypotheses less restrictive than those by Blasius (variance of pressure, negligible curvature of trajectory of the fluid particles).

After an extremely short transition, a state of slow variation is established where the friction depends little on x_j, for which reason this state is designated as "asymptotic solution". The friction develops with the Reynolds number in a manner quite close to the experimental turbulent friction.

As indicated above, the presence of the sublayer is connected with strong rotation components along the wall and with high friction. The configuration of the velocity field is close to that of the mean velocity fields of a turbulent boundary layer, except that several elements are lacking for a complete solution (although of only an approximate type), specifically the means for determining the point x_j at which this phenomenon starts. The previously demonstrated transition extends over lengths infinitely shorter than those experimentally observed. Similarly, no explanation has ever been furnished for the violently nonstationary state which led to the designation "turbulent flow".

However, for $\frac{\partial U_0}{\partial x} \cong 0$, nothing in the Blasius method permits detecting a cause for the change of state in the stationary regime.

It will be noted that the Blasius hypothesis (invariance of pressure in the laminar boundary layer) is compatible only with large radii of curvature of the particle trajectories and thus with limited intensities of rotation.

This state of affairs obviously can stop existing in the presence of a nonstationary perturbation brought in from outside. This returns us to the notation used by Schlichting in his method of perturbations; here, the results to which this method might lead will be examined in some detail.

This represents the first object of Part II of this exposé.

To facilitate the analysis, we first attempted to represent the stationary Blasius solution U(Y) by an expansion in powers of the type

$$U(Y) = U_0 \left[1 - \sum_i a_i e^{-\alpha_i \sqrt{\frac{U_0}{\nu x}} \cdot Y} \right]$$

where $\sum_i a_i = 1$.

The next step was to take a perturbation of the external field of the simplest possible type which, in the axes fixed with respect to this exterior space reduces to a small normal component (in v) harmonic to x and t. When transferred into the axes fixed to the wall, this generates a nonstationary flow /84 with sinusoidal streamlines in the field outside the boundary layers.

It is then sufficient to write the Navier equations and, retaining the principal terms, to apply the proper boundary conditions after elimination of the pressure, so as to study the fate of the perturbation within the boundary layer.

This more or less represents the calculation method already used in Part I.

It will be shown that, beyond a critical segment x_c, the pressure will no longer be invariant in the boundary layer. The laminar state, based on the invariance of the pressure, can thus no longer subsist and must make room for the second type of flow, studied in Part I under the designation of "stationary turbulent" flow, which is compatible with the variance of pressure.

In the following Chapters, the nature and propagation of a tangential velocity perturbation (in u) at the interior of the boundary layer (laminar and "turbulent") will be investigated. An application of the results permits demonstrating the real transition and, downstream from this, the existence of non-stationary components of permanent state, explaining the reason for calling such a flow "turbulent".

CHAPTER IV /85

EFFECT OF A HARMONIC EXTERIOR PERTURBATION

23. Introduction of a Harmonic Perturbation in v'; Axes Fixed in Space

Let us consider axes XY fixed with respect to the exterior space (zero mean velocity) and stationary perturbation velocity components u', v' which are zero in the spatial mean* and also are small; this will permit - since ν itself is small - to neglect the corresponding viscous terms in first analysis.

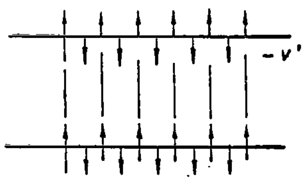

Fig.18

The Euler equations are written in the form

$$u'\frac{\partial u'}{\partial X} + v'\frac{\partial u'}{\partial Y} + \frac{1}{\rho}\cdot\frac{\partial p'}{\partial X} = 0, \qquad u'\frac{\partial v'}{\partial X} + v'\frac{\partial v'}{\partial Y} + \frac{1}{\rho}\cdot\frac{\partial p'}{\partial Y} = 0.$$

The continuity condition will be

$$\frac{\partial u'}{\partial X} + \frac{\partial v'}{\partial Y} = 0.$$

To satisfy this condition, u', v' will be derived from a stream function for which we will use the form $\psi(X, Y) = \varphi(Y) \cdot f(X)$:

$$u' = \frac{\partial \psi}{\partial Y} = \varphi'_Y \cdot f, \qquad v' = -\frac{\partial \psi}{\partial X} = -\varphi \cdot f'_X.$$

From this, we obtain the equations

$$\varphi'_Y \cdot f \cdot \varphi'_Y \cdot f'_X - \varphi \cdot f'_X \cdot \varphi''_{Y^2} \cdot f + \frac{1}{\rho}\cdot\frac{\partial p'}{\partial X} = 0,$$

$$-\varphi'_Y \cdot f \cdot \varphi \cdot f''_{X^2} + \varphi \cdot f'_X \cdot \varphi'_Y \cdot f'_X + \frac{1}{\rho}\cdot\frac{\partial p'}{\partial Y} = 0.$$

Let us eliminate p' by deriving the first equation with respect to Y and the second with respect to X, and by then subtracting therefrom /86

$$2\varphi'_Y \cdot \varphi''_{Y^2} \cdot ff'_X - (\varphi'_Y \varphi''_{Y^2} + \varphi\varphi'''_{Y^3})ff'_X + \varphi\varphi'_Y (f'_X f''_{X^2} + ff'''_{X^3}) - 2\varphi\varphi'_Y (f''_{X^2} f'_X) = 0,$$

* For u' ≡ 0, a scheme in v' can be conceived which would obey this definition, by assuming the space in question as placed between two horizontal walls over which regularly intercalated sources and sinks of sinusoidal intensity distribution are distributed (Fig.18).

i.e.,

$$ff'_X(\varphi'_Y\varphi''_{Y^2}-\varphi\varphi'''_{Y^3})-\varphi\varphi'_Y(f'_X f''_{X^2}-ff'''_{X^3})=0.$$

The integration conditions read

$$\frac{\varphi'''_{Y^3}}{\varphi''_{Y^2}}=\frac{\varphi'_Y}{\varphi},\qquad \frac{f'''_{X^3}}{f''_{X^2}}=\frac{f'_X}{f}.$$

Thus,

$$\text{Log}\,\varphi''_{Y^2}=\text{Log}\,\varphi+\text{Log}\,C^2,\quad \text{where } C^2=\text{const},$$

i.e.,

$$\varphi''_{Y^2}-C^2\varphi=0.$$

With $C = \mathring{i}a$

$$\varphi=\varphi_1 e^{\mathring{i}aY}+\varphi_2 e^{-\mathring{i}aY}=\varphi_0\cos aY,$$

we obtain a solution satisfying the above conditions.

Similarly for f, we have

$$\text{Log}\,f''_{X^2}=\text{Log}\,f+\text{Log}\,\gamma^2.$$

where $\gamma = \mathring{i}\alpha$,

$$f=f_0\cos\alpha X.$$

Then,

$$u'=-f_0\varphi_0 a\sin aY\cdot\cos\alpha X,$$

$$v'=f_0\varphi_0\alpha\cos aY\cdot\sin\alpha X.$$

The streamline is such that

$$Y'_X=\frac{v'}{u'}=-\frac{\alpha}{a}\tan aY\cdot\cot\alpha X.$$

A particular solution is any $\varphi'_Y = 0$, $\varphi = \text{const} \cdot \varphi_0$, so that

$$u'=0;\qquad \frac{\partial v'}{\partial Y}=0,\qquad v'=\varphi_0 f_0\alpha\sin\alpha x.$$

We will restrict our study to this case.

24. Axes Fixed in the Plate /87

Let us now consider a plane plate with a velocity $|U_0|$ in this space. In the axes xy, fixed with respect to the wall of this plate (Fig.19), we have

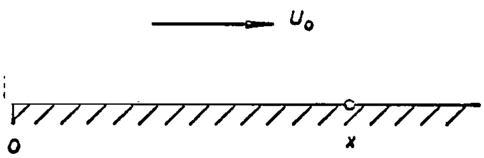

Fig.19

$$\begin{cases} X = x - U_0 t, \\ Y = y. \end{cases}$$

$$\frac{\partial X}{\partial x} = 1, \qquad \frac{\partial Y}{\partial y} = 1, \qquad \frac{\partial X}{\partial t} = -U_0, \qquad \frac{\partial Y}{\partial t} = 0,$$

$$f'_x = f'_X \cdot X'_x = f'_X, \qquad f'_t = f'_X \cdot X'_t = -U_0 f'_X,$$

$$f(X) = f(x - U_0 t) = f(x_1 t).$$

The Euler equations, written in the domain exterior to the boundary layer when reduced to the principal terms (U_0 large with respect to u' and v'), will yield

$$\frac{\partial U}{\partial t} + U_0 \frac{\partial U}{\partial x} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial x} = 0 \qquad \text{or} \qquad \varphi'_y f'_t + U_0 \varphi'_y f'_x + \frac{1}{\rho} \cdot \frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial V}{\partial t} + U_0 \frac{\partial V}{\partial x} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial y} = 0 \qquad \text{or} \qquad -\varphi f''_{xt} - U_0 \varphi f''_{x^2} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial y} = 0.$$

For the solution $u' = 0$, the following remains

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0 \qquad \text{since} \qquad \varphi'_y = 0 \quad \text{and} \quad f''_{xt} = -U_0 f''_{x^2}.$$

This means that φ_0 and f can be arbitrary values, without introducing any contradiction as to the external flow.

Consequently, an arbitrary constant $\varphi = \varphi_0$ and

$$\varphi_0 f = g = g_0 \cos \alpha (x - U_0 t)$$

can be taken into consideration.

25. Introduction of Dissipative Navier Terms

The Navier-Stokes equations, in the axes fixed to the wall, are written in the form

$$\varphi'_y \cdot f'_t + U_0 \varphi'_y f'_x + \frac{1}{\rho} \cdot \frac{\partial p}{\partial x} = \nu [\varphi'_y \cdot f''_{x^2} + \varphi'''_{y^3} f],$$

$$-\varphi \cdot f''_{xt} - U_0 \varphi f''_{x^2} + \frac{1}{\rho} \cdot \frac{\partial p}{\partial y} = -\nu [\varphi f'''_{x^3} + \varphi''_{y^2} f'_x].$$

From this, by elimination of p, we obtain

$$\varphi''_{y^2} \cdot f'_t + U_0 \varphi''_{y^2} f'_t + \varphi f'''_{x^2 t} + U_0 \varphi f'''_{x^3} = \nu [\varphi''_{y^2} f''_{x^2} + \varphi''''_{y^4} f + \varphi f''''_{x^4} + \varphi''_{y^2} f''_{x^2}],$$

i.e.,

$$\varphi''_{y^2}[f'_t + U_0 f'_x] + \varphi\,[f'''_{x^2 t} + U_0 f'''_{x^3}] = \nu\,[\varphi''''_{y^4} f + 2\,\varphi''_{y^2} f''_{x^2} + \varphi\, f''''_{x^4}].$$

The solution

$$\varphi''_{y^2} - C^2 \varphi = 0 \quad \text{yields} \quad \varphi = \varphi_1 e^{Cy} + \varphi_2 e^{-Cy},$$

$$f_{x^2} - \gamma^2 f = 0 \quad \text{yields} \quad f = f_1 e^{\gamma x} + f_2 e^{-\gamma x}.$$

Substituting this in the general equation, we obtain the condition

$$C^2 \{ f'_t + U_0 f'_x \} + f'''_{x^2 t} + U_0 f'''_{x^3} = \nu\,[C^4 f + 2\,C^2 f''_{x^2} + f''''_{x^4}],$$

i.e., for the term in $f_1 e^{\gamma x}$,

$$C^2 (f'_{1t} + U_0 \gamma f_1) + f'_{1t} \cdot \gamma^2 + U_0 \gamma^3 f_1 = \nu\,[C^4 f_1 + 2\,C^2 \gamma^2 f_1 + \gamma^4 f_1]$$

or

$$f'_{1t}[C^2 + \gamma^2] + f_1 [U_0 \gamma (C^2 + \gamma^2) - \nu\,(C^2 + \gamma^2)^2] = 0,$$

$$\frac{f'_{1t}}{f_1} = -[U_0 \gamma - \nu\,(C^2 + \gamma^2)],$$

so that, finally,

$$f_1 = f_1\, e^{-[U_0\gamma - \nu(C^2+\gamma^2)]t}\ .$$

Similarly,

$$f_2 = f_{2_0}\, e^{[U_0\gamma + \nu(C^2+\gamma^2)]t}\ .$$

Consequently, the solution f is written in the form /89

$$f = [f_{1_0} e^{\gamma(x - U_0)t} + f_{2_0} e^{-\gamma(x - U_0 t)}]\, e^{\nu(C^2+\gamma^2)t}.$$

If γ is imaginary $\gamma = i\alpha$, we obtain

$$f = [f_{0_0} e^{i\alpha(x - U_0 t)} + f_{2_0} e^{-i\alpha(x - U_0 t)}]\, e^{\nu(C^2-\alpha^2)t}.$$

The solution will be maintained if $C^2 - \alpha^2 = 0$, whence

$$\varphi''_{y^2} - \alpha^2 \varphi = 0 \quad \text{and} \quad \varphi = \varphi_1 e^{\alpha y} + \varphi_2 e^{-\alpha y} \quad \text{in real expotentials.}$$

26. Generation and Propagation of the External Perturbation

Let us assume that φ has the following form:

$$\varphi = \varphi_1 e^{\alpha y} + \varphi_2 e^{-\alpha y} + \varphi_0,$$

where φ_0 is exclusively a function $\varphi_0(t)$ of time. Then, coming from φ_0,

$$u' = \varphi'_y f = 0, \qquad v' = -\varphi_0 f'_x, \qquad \frac{\partial u'}{\partial t} = 0, \qquad \frac{\partial v'}{\partial t} = -\varphi'_{0t} \cdot f'_x.$$

The term φ_0 will appear only in the second Navier-Stokes equation, over a contribution

$$-\varphi'_{0t} \cdot f'_x - \varphi_0 f''_{xt} - U_0 \varphi_0 f''_{x^2} \quad \text{to the first term}$$

and

$$-\nu \varphi_0 f'''_{x^3} \quad \text{to the second.}$$

In the equation for the elimination of pressure, this term will correspond to

$$\varphi'_{0t} f''_{x^2} + \varphi_0 f'''_{x^2t} + U_0 \varphi_0 f'''_{x^3} = \nu \varphi_0 f''''_{x^4}$$

For $f = f_1, \; e^{\mathring{i}\alpha(x-U_0t)}$, we obtain

$$-\varphi'_{0t} \alpha^2 + \varphi_0 U_0 \mathring{i} \alpha^3 - U_0 \varphi_0 \mathring{i} \alpha^3 = \nu \varphi_0 \alpha^4.$$

This leaves /90

$$\frac{\varphi'_{0t}}{\varphi_0} = -\nu \alpha^2 \quad \text{whence} \quad \varphi_0 = \varphi_{00} e^{-\nu\alpha^2 t}$$

and the same solution for $f_{20} e^{-\mathring{i}\alpha(x U_0 t)}$.

Since $\frac{\nu\alpha^2}{\alpha U_0}$ is very small, this means introduction of a term φ_0 varying very slowly in time with respect to f.

If one desires to maintain the motion in $\varphi_0 f_{1_0}$, it is necessary - during each period of time dt $\bar{z}$ to introduce constantly from the exterior a component $d\varphi_0$ equal to $\nu\alpha^2\varphi_{00} e^{-\nu\alpha^2 t} \cdot dt$ which replaces the dissipated component.

No matter how this might be, propagation of the component $\varphi_0 f_{1_0}$ takes place at a velocity U_0. If this is generated at a point x^* very far upstream, for example at the time $t = 0$, it will arrive at x at the time $t = \frac{x - x^*}{U_0}$ and its state will be characterized, with respect to the initial state, by an attenuation such that

$$[\varphi_0]_x = \varphi_{0_3} e^{-\nu\alpha^2 \cdot \frac{x-x^*}{U_0}}$$

If, at each instant following the initial time t = 0, there is generated in x^* a new perturbation such that

$$\varphi_{00}(t) = \int_0^t \varphi'_{00_t} \cdot dt = \varphi'_{00_t} \cdot t \qquad (\varphi'_{00_t} = \text{const}),$$

then the law of attenuation, in x, will lead to

$$\varphi'_{00_t} \int_{t=\frac{x-x^*}{U_0}}^{t} e^{-\nu\alpha^2 t} dt = \varphi'_{00_t} \frac{1}{\nu\alpha^2} \left[e^{-\nu\alpha^2 \frac{x-x^*}{U_0}} - e^{-\nu\alpha^2 t} \right] = \varphi_0(x, t), \qquad t > \frac{x-x^*}{U_0}.$$

When $t \to \infty$, a limited stationary state will remain in x, characterized by

$$\varphi_0(x)_\infty = \varphi'_{00_t} \cdot \frac{1}{\nu\alpha^2} e^{-\nu\alpha^2 \cdot \frac{x-x^*}{U_0}}.$$

Thus, the introduction - into an initial permanent state - of a perturbation, produced at an upstream point x^* and maintained linearly as a function of time, will lead - at a downstream point x - to progressive establishment of this attenuated perturbation starting from a time shifted by $\frac{x - x^*}{U_0}$. The permanent limiting state toward which tends the perturbation in x is such that (Fig. 20)

$$\varphi_0(x, t) = \varphi'_{00_t} \frac{1}{\nu\alpha^2} e^{-\nu\alpha^2 \cdot \frac{x-x^*}{U_0}}.$$

To this there corresponds $\frac{\partial v'}{\partial t} = 0$ but, since $x = x^*$ is large with respect to x^*, also $\frac{\partial v'}{\partial x} \cong 0 \left(\text{and } \frac{\partial v'}{\partial y} = 0\right)$.

For a sinusoidal perturbation in x^*, we will also have a limit sinusoidal response attenuated in x (Fig. 21).

27. Nonstationary Perturbation Interior to the Boundary Layer; General Equations /92

It is here a question of writing the Navier-Stokes equation for the variations introduced by the presence of the perturbations u', v'.

Let us take a function ψ of the stream relative to these perturbations, having the form

$$\psi = \varphi(y) \cdot f(x, t) + g(x, t),$$

such that

$$u' = \frac{\partial \psi}{\partial y} = \varphi_y' \cdot f, \qquad v' = -\frac{\partial \psi}{\partial x} = -\varphi f'_x - g'_x;$$

$$\frac{\partial u'}{\partial t} = \varphi'_y f'_t, \qquad \frac{\partial v'}{\partial t} = -\varphi f''_{xt} - g''_{xt};$$

$$\frac{\partial u'}{\partial x} = -\varphi'_y f'_x, \qquad \frac{\partial v'}{\partial x} = -\varphi f''_{x^2} - g''_{x^2}, \qquad \frac{\partial u'}{\partial y} = \varphi''_{y^2} f, \qquad \frac{\partial v'}{\partial y} = -\varphi'_y f'_x;$$

$$\frac{\partial^2 u'}{\partial x^2} = -\varphi'_y f''_{x^2}, \qquad \frac{\partial^2 v'}{\partial x^2} = -\varphi f'''_{x^3} - g'''_{x^3}, \qquad \frac{\partial^2 u'}{\partial y^2} = \varphi'''_{y^3} f, \qquad \frac{\partial^2 v'}{\partial y^2} = -\varphi''_{y^2} f'_x.$$

/91

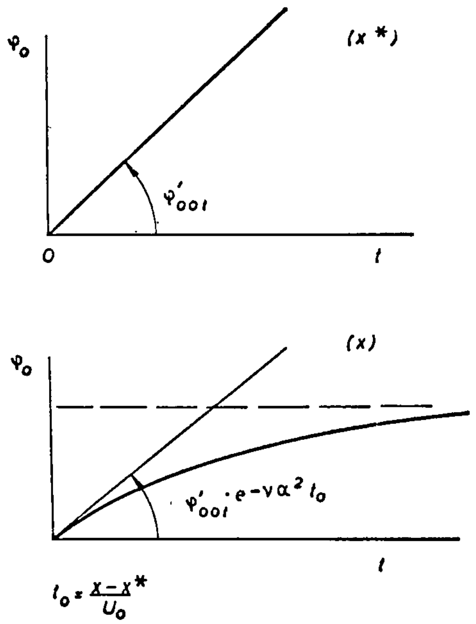

Fig.20

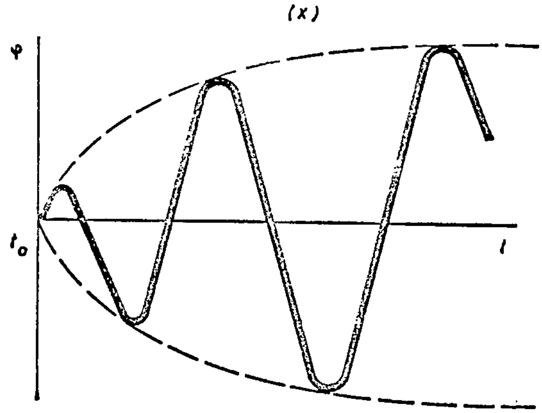

Fig.21

From this follow the two Navier equations in the principal terms

$$\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{\partial U}{\partial y} + \frac{1}{\rho} \cdot \frac{\partial p'}{\partial x} = \nu \Delta^2 u',$$

$$\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} + \frac{1}{\rho} \cdot \frac{\partial p'}{\partial y} = \nu \Delta^2 v',$$

where U is the pre-existing velocity, namely

$$\varphi_y' f'_t + U \varphi'_y f'_x - (\varphi f'_x + g'_x) U'_y + \frac{1}{\rho} \frac{\partial p'}{\partial x} = \nu [\varphi'_y f''_{x^2} + \varphi'''_{y^3} f],$$

$$-(\varphi f''_{xt} + g''_{xt}) - U (\varphi f''_{x^2} + g''_{x^2}) + \frac{1}{\rho} \cdot \frac{\partial p'}{\partial y} = -- \nu [\varphi f'''_{x^3} + \varphi''_{y^2} f + g'''_{x^3}].$$

Elimination of the pressure leads to

$$\varphi''_{y^2}[f'_t + Uf'_x] + \varphi[\,\}f'''_{x^2t} + Uf'''_{x^3}\} + \{f''_{x^2}\,U'_x\{ - U''_{y^2}f'_x]$$
$$+ [g''_{x^2t} + Ug'''_{x^3} + \{U'_x\,g''_{x^2}\} - U''_{y^2}\,g'_x] = \nu\,[f\,\varphi''''_{y^4} + 2\,\varphi''_{y^2}f''_{x^2} + \varphi\,f''''_{x^4} + g''''_{x^4}].$$

The terms in U'_x, forming the product with other derivatives in x, can be neglected.

Two different cases will be studied here: /93

1) that of the laminar layer where

$$U = U_0\left[1 - \sum_i a_i\,e^{-\alpha_i y\sqrt{U_0/\nu x}}\right] \quad \text{and} \quad U''_{y^2} = -U_0\sum_i a_i\,\alpha_i^2\frac{U_0}{\nu x}\,e^{-\alpha_i y\sqrt{U_0/\nu x}}.$$

2) that of the "stationary turbulent" layer whose field U(y) has been defined above (Part I).

28. Reaction of the Laminar Boundary Layer to the Harmonic External Perturbation in v'

Within the frame of the Blasius theory, no interface exists theoretically between the laminar boundary layer and the exterior flow U_0.

Compared to the tangential velocity component U, as already emphasized, the term $\frac{U_0 - U}{U_0}$, for $y = \delta_L \cong 5.5\sqrt{\frac{\nu x}{U_0}}$ becomes so small that the question assumes an entirely academic character.

This is not at all the same with respect to the normal component v.

28.1 Upper Border of the Laminar Boundary Layer

It is of some use to dwell on this item.

As is known, the pertaining theory stipulates first - over the approximations and hypotheses used - that the pressure be invariant in the boundary layer. It then requires determination of an auxiliary stream function $f(\eta)$ by means of the well-known third-order equation $ff''_{\eta^2} + 2f'''_{\eta^3} = 0$; three boundary conditions are connected with this which, with

$$\eta = y\sqrt{\frac{U_0}{\nu x}}, \qquad U = U_0\cdot f'_\eta, \qquad V = \frac{1}{2}\sqrt{\frac{\nu U_0}{x}}\,(\eta f' - f),$$

are the following:

$$U(0) = 0, \qquad V(0) = 0, \qquad U(\infty) = U_0.$$

We thus find

$$f'_\eta(\infty) = 1, \qquad f''_{\eta^2}(\infty) = 0$$

where f'_η actually rejoins the asymptote 1 for $\eta = 5.5$, where $1 - f'_\eta \cong 0.005$.

Let η_0 be a value of η such that $f'_\eta \cong 1$ (for example, $\eta_0 = 5.5$).

Let us put

$$\varepsilon'_\eta(\eta) = 1 - f'_\eta(\eta)$$

where $\varepsilon'_\eta(\eta)$ is very small when $\eta > \eta_0$.

Since /94

$$f_\eta(\eta) = \int_0^{\eta_0} f'_\eta \, d\eta + \int_{\eta_0}^{\eta} (1 - \varepsilon'_\eta) \, d\eta = f(\eta_0) + |\eta - \varepsilon|_{\eta_0}^{\eta} = f_0 + \Delta\eta - \Delta\varepsilon,$$

where f_0 represents $f(\eta_0)$, $\Delta\eta - \eta_0$, $\Delta\varepsilon = \varepsilon(\eta) - \varepsilon_0$.

However, $f''_{\eta^2} = -\varepsilon''_{\eta^2}$, $f'''_{\eta^3} = -\varepsilon'''_{\eta^3}$. Then, the equation of the third order in the domain $\eta > \eta_0$ is written as

$$(f_0 + \eta - \eta_0 - \varepsilon + \varepsilon_0)\,\varepsilon''_{\eta^2} + 2\,\varepsilon'''_{\eta^3} = 0,$$

i.e., neglecting the terms of the second infinitesimal order,

$$\frac{\varepsilon'''_{\eta^3}}{\varepsilon''_{\eta^2}} = -[f_0 - \eta_0 + \eta]\,\frac{1}{2},$$

or else

$$\operatorname{Log} \varepsilon''_{\eta^2}(\eta) = -\frac{1}{2}\left[(f_0 - \eta_0)\,\eta + \frac{\eta^2}{2}\right] + \operatorname{Log} A,$$

where A is a constant.

Thus,

$$\varepsilon''_{\eta^2}(\eta) = A e^{-\left[\frac{\eta}{2}(f_0 - \eta_0) + \frac{\eta^2}{4}\right]}$$

so that we have

$$\varepsilon''_{\eta^2}(\eta_0) = A e^{-\frac{1}{2}\left[\eta_0 f_0 - \eta_0^2 \frac{1}{2}\right]}$$

or else

$$\varepsilon''_{\eta^2}(\eta) = \varepsilon''_{\eta^2}(\eta_0) \cdot e^{-\frac{1}{2}\left[(f_0 - \eta_0)(\eta - \eta_0) + \frac{\eta^2 - \eta_0^2}{2}\right]},$$

whence, finally,

$$f''_{\eta^2}(\eta) = f''_{\eta^2}(\eta_0) \cdot e^{-\frac{1}{2}\left[(f_0 - \eta_0)(\eta - \eta_0) + \frac{\eta^2 - \eta_0}{2}\right]}.$$

To study the development of $V = \frac{1}{2}\sqrt{\frac{\nu U_0}{x}}\,[\eta f'_\eta - f]$ for $\eta > \eta_0$, since

$$V(0) = 0,$$

we must form

$$\int_{\eta_0}^{\eta} \frac{d}{d\eta}[\eta f'_\eta - f]\, d\eta = \int_{\eta_0}^{\eta} \eta f''_{\eta^2}\, d\eta = \int_{\eta_0}^{\eta} f''_{\eta^2}(\eta_0) \cdot \eta\, e^{-\frac{1}{2}\left[(f_0 - \eta_0)(\eta - \eta_0) + \frac{\eta^2 - \eta_0^2}{2}\right]} \cdot d\eta.$$

However, $f''_{\eta^2}(\eta_0) > 0$. Consequently, this integral is not zero: It increases constantly with increasing η so that, since $V(\eta_0) > 0$ is nonzero, we have

$$V(\infty) > V(\eta_0) > 0.$$

Although there is nothing that contradicts obtaining $V(\eta_0) \neq 0$, this is not so for $V(\infty) \neq 0$ which (angularly) alters the orientation of the velocity at $y = \infty$, in contrast to the initial data (and alters it even more than at $\eta = \eta_0$). /95

Thus, to avoid this contradiction, one is forced to limit the expansion of η to a finite value η_0 which, apparently, is arbitrary but - for the reasons given above - must be fixed in the domain $5 < \eta_0 < 6$. (Prandtl gave a value of 5.2 and we took 5.5.) The minor discrepancies resulting from this must be attributed to the computational uncertainties produced by the approximations used in the theory, which latter nevertheless is quite efficient and faithful to known facts, but can be so only under the condition that a limit η_0 is placed on its application domain.

Thus, in the logical exploitation of the theory, one is forced to admit the existence of an effective border of the laminar layer.

This digression was justified since we now must investigate the connectivity conditions of applied external perturbations as well as the reaction of the laminar boundary layer to these.

28.2 Equations Defining Perturbation in the Laminar Layer

Let there be a state of flow exterior to the laminar boundary layer such that $U = U_0$ = const and let there be a harmonic perturbation whose stream function is

$$\psi_{ext} = \varphi(y)\, f(x, t) + g(x, t),$$

where f and g are pulsation functions α with respect to x and U_0t, while φ is a function of the general form given above for the (stationary) perturbations, $\varphi = \varphi_1 e^{\alpha y} + \varphi_2 e^{-\alpha y}$ (see Sect.25):

$$f = f_0 \cos \alpha (x - U_0 t), \qquad g = g_0 \cos \alpha (x - U_0 t).$$

These respond to

$$f'''_{x^3} + \alpha^2 f'_x = 0, \qquad g'''_{x^3} + \alpha^2 g'_x = 0, \qquad \varphi''_{y^2} - \alpha^2 \varphi = 0,$$

and finally, to

$$f'_t + U_0 f'_x = 0, \qquad g'_t + U_0 g'_x = 0 \quad \text{and} \quad g'''_{x^3t} + U_0 g'''_{x^3} = 0.$$

The general conditions such as (see Sect.25)

$$\varphi''_{y^2} (f'_t + U_0 f'_x) + \varphi (f'''_{x^3t} + U_0 f''''_{x^4}) = 0$$

are satisfied; in particular, we have

$$\frac{\varphi''_{y^2}}{\varphi} = \alpha^2 = -\frac{f'''_{x^3}}{f}$$

$\Big[$i.e., $\varphi''_{y^2} f'_x + f'''_{x^3}\varphi = 0$ which is $\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}\right) = -2\frac{\partial \omega'}{\partial x} = 0$ and thus reflects the conservation of rotation$\Big]$.

At the interior of the laminar layer, for which the Blasius law will be taken in the form of /96

$$U = U_0 \left[1 - \sum_i a_i e^{-\alpha_i \sqrt{\frac{U_0}{\nu x}} y}\right],$$

we obtain the following for the fundamental equation derived by Navier (see Sect.27):

$$\varphi''_{y^2} (f'_t + U f'_x) + \varphi [(f'''_{x^3t} + U f'''_{x^3}) - U''_{y^2} f'_x] + [g'''_{x^3t} + U g'''_{x^3} - U''_{y^2} g'_x] = 0,$$

i.e., first of all

$$\varphi''_{y^2} (f'_t + U_0 f'_x) + \varphi [f'''_{x^3t} + U_0 f'''_{x^3}] + g'''_{x^3t} + U_0 g'''_{x^3} = 0$$

or else

$$\left\{\begin{aligned} f'_t + U_0 f'_x &= 0, \\ (g'_t + U_0 g'_x)''_{x^2} &= 0. \end{aligned}\right. \qquad \text{(A)}$$

Let us set

$$\Phi(\eta) = \sum_i a_i e^{-\alpha_i \eta} \qquad \text{with} \qquad \eta = \sqrt{\frac{U_0}{\nu x}}\, y,$$

$$U = U_0[1-\Phi(\eta)], \qquad U''_{y^2} = -U_0\, \Phi''_{\eta^2}\cdot\frac{U_0}{\nu x}.$$

On the other hand, we obtain

$$\Phi(\eta)\left[\varphi''_{y^2} f'_x + \varphi\left\{f'''_{x^3} - \frac{\Phi''_{\eta^2}}{\Phi}\cdot\frac{U_0}{\nu x} f' x\right\} + g'''_{x^3} - \frac{\Phi''_{\eta^2}}{\Phi}\cdot\frac{U_0}{\nu x} g'_x\right] = 0,$$

whence

$$\varphi''_{y^2} f'_x + \varphi\left(f'''_{x^3} - \frac{\Phi''_{\eta^2}}{\Phi}\cdot\frac{U_0}{\nu x} f'_x\right) = 0, \qquad g'''_{x^3} - \frac{\Phi''_{\eta^2}}{\Phi}\cdot\frac{U_0}{\nu x} g'_x = 0. \tag{B}$$

Equations (A) known as "time equations" will be satisfied if the solutions f and g are expanded in $x - U_0 t$ with respect to x and t.

From eqs.(B) it follows that f and g will be functions of x alone (at the exclusion of y), in agreement with the expansion adopted for the perturbation function

$$\psi(x, y, t) = \varphi(y)\cdot f(x, t) + g(x, t),$$

in the single case in which $\frac{\Phi''_{\eta^2}}{\Phi}$ reduces to a constant relative to η (and thus to y). This will also be the only case for which the problem can be solved.

However, $\Phi(\eta) = \sum_i a_i e^{-\alpha_i \eta}$ is derived directly from the function $f(\eta)$ of /97
the laminar Blasius field over $f'_\eta(\eta) = 1 - \Phi(\eta)$. Thus, this function is given. It is then obvious that, with an expansion in three terms such as

$$\alpha_3 = \overline{\alpha}_i, \qquad \alpha_{1,2} = \overline{\alpha}_i \pm \Delta\alpha_i = \overline{\alpha}_i(1 \pm \epsilon)$$

where ϵ is small, it becomes possible

to define the development of the Blasius function with respect to η (and y);

to satisfy, to within terms of the second order in ϵ^2, the condition

$$\frac{\Phi''_{\eta^2}}{\Phi} = \text{const}\ \overline{\alpha_i^2}$$

(see Appendix 1), while still closely obeying the Blasius law, as indicated in Diagram I of Section 21.7.

This means that the decomposition adopted for the stream function ψ will furnish a very approximate (but not exact) picture of the real perturbation. It then becomes possible to put

$$f = f_{ext} + \delta f, \qquad g = g_{ext} + \delta g,$$

where $\delta f(x, t)$, $\delta g(x, t)$ are the variations of these functions appearing between the outer and inner domain at the laminar boundary layer.

Since $f_{ext} = f_0 \cos \alpha (x - U_0 t)$, $g_{ext} = g_0 \cos \alpha (x - U_0 t)$, it follows that

$$\varphi''_{y^2} f'_{ext_x} + \varphi f'''_{ext_{x^3}} + \varphi''_{y^2} \delta f'_x + \varphi \delta f'''_{x^3} - \varphi \overline{\alpha}_i^2 \frac{U_0}{\nu x} (f'_{x_{ext}} + \delta f'_x) = 0,$$

i.e.,

$$\varphi''_{y^2} \delta f'_x + \varphi\left(\delta f'''_{x^3} - \overline{\alpha}_i^2 \frac{U_0}{\nu x} \delta f'_x\right) + \left[\varphi''_{y^2} - \left(\alpha^2 + \overline{\alpha}_i^2 \frac{U_0}{\nu x}\right)\varphi\right] f'_{x_{ext}} = 0.$$

Similarly,

$$\delta g'''_{x^3} - \overline{\alpha}_i^2 \frac{U_0}{\nu x} \delta g'_x - \left(\alpha^2 + \overline{\alpha}_i^2 \frac{U_0}{\nu x}\right) g'_{x_{ext}} = 0.$$

Since φ is independent of x (and of t), it is necessary that

$$\frac{\varphi''_{y^2}}{\varphi} = -\frac{\delta f'''_{x^3} - \overline{\alpha}_i^2 \frac{U_0}{\nu x} \delta f'_x - \left(\alpha^2 + \overline{\alpha}_i^2 \frac{U_0}{\nu x}\right) f'_{ext_x}}{f'_{x_{ext}} + \delta f'_x} = k^2$$

is constant (with respect to x and y) or /98

$$\delta f'''_{x^3} + \left(k^2 - \overline{\alpha}_i^2 \frac{U_0}{\nu x}\right) \delta f'_x + f'_{x_{ext}}\left(k^2 - \overline{\alpha}_i^2 \frac{U_0}{\nu x} - \alpha^2\right) = 0$$

and

$$\varphi = \varphi_1 e^{ky} + \varphi_2 e^{-ky}.$$

28.3 <u>Solution</u>

The equations in δf and δg can be solved only for a calculation in steps.

Thus, let us consider the origin x_0 of one of these. Let us execute the step Δx; for the condition in δf, expressing the variations of f_{ext} and the quantities in x, we have

$$\delta f'''_{\Delta x^3} + \delta f'_{\Delta x}\left(k^2 - \overline{\alpha}_i^2 \frac{U_0}{\nu x_0}\right) + f_0 \alpha \sin \alpha (x_0 - U_0 t)\left(\overline{\alpha}_i^2 \frac{U_0}{\nu x_0} + \alpha^2 - k^2 - \overline{\alpha}_i^2 \frac{U_0}{\nu x_0} \cdot \frac{\Delta x}{x_0}\right)$$
$$+ f_0 \alpha^2 \cos \alpha (x_0 - U_0 t)\left(\overline{\alpha}_i^2 \frac{U_0}{\nu x_0} + \alpha^2 - k^2\right) \Delta x = 0.$$

Let us put

$$\beta = \sqrt{\overline{\alpha}_i^2 \frac{U_0}{\nu x_0} - k^2}.$$

The solution of $\delta f'_x$ will have the form

$$\delta f'_x = \beta\,[\delta F_1\, e^{\beta(\Delta x - U_0 t)} - \delta F_2\, e^{-\beta(\Delta x - U_0 t)}] + b + 2\,c\,\Delta x.$$

Fixing the time t under consideration, we can write

$$\delta f'_{\Delta x} = \beta\,[\delta f_1\, e^{\beta \Delta x} - \delta f_2\, e^{-\beta \Delta x}] + b + 2\,c\,\Delta x.$$

Substituting this in the above general equation, we obtain

$$\left(k^2 - \overline{\alpha_i^2}\frac{U_0}{\nu x_0}\right)(2\,c\,\Delta x + b) + f_0\Big[\alpha \sin\alpha\,(x_0 - U_0\,t)\cdot\left(\overline{\alpha_i^2}\frac{U_0}{\nu x_0} + \alpha^2 - k^2\right)$$
$$- \Delta x\left\{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}\frac{\alpha}{x_0}\sin\alpha\,(x_0 - U_0\,t) - \alpha^2\cos\alpha\,(x_0 - U_0\,t)\left(\overline{\alpha_i^2}\frac{U_0}{\nu x_0} + \alpha^2 - k^2\right)\right\}\Big] = 0,$$

whence:

$$b\left(k^2 - \overline{\alpha_i^2}\frac{U_0}{\nu x_0}\right) + f_0\,\alpha\sin\alpha\,(x_0 - U_0\,t)\cdot\left(\overline{\alpha_i^2}\frac{U_0}{\nu x_0} + \alpha^2 - k^2\right) = 0,$$

$$2\,c\left(k^2 - \overline{\alpha_i^2}\frac{U_0}{\nu x_0}\right) - f_0\left\{\alpha_i^2\frac{U_0}{\nu x_0}\cdot\frac{\alpha}{x_0}\sin\alpha\,(x_0 - U_0\,t)\right.$$
$$\left. - \alpha^2\cos\alpha\,(x_0 - U_0\,t)\left(\overline{\alpha_i^2}\frac{U_0}{\nu x_0} + \alpha^2 - k^2\right)\right\} = 0,$$

which determines b and c (constant within one step but variable from step to step).

It is immediately obvious that, if $k^2 = \alpha^2$, /99

$$b = -\frac{\overline{\alpha_i^2}\dfrac{U_0}{\nu x_0}}{\alpha^2 - \overline{\alpha_i^2}\dfrac{U_0}{\nu x_0}}\, f_0\,\alpha\sin\alpha\,(x_0 - U_0\,t),$$

$$2\,c = \frac{\overline{\alpha_i^2}\dfrac{U_0}{\nu x_0}}{\alpha^2 - \overline{\alpha_i^2}\dfrac{U_0}{\nu x_0}}\, f_0\left[\frac{\alpha}{x_0}\sin\alpha\,(x_0 - U_0\,t) - \alpha^2\cos\alpha\,(x_0 - U_0\,t)\right].$$

Finally, the solution in δg will be of similar appearance, characterized by

$$\delta g'''_{\Delta x^3} - \overline{\alpha_i^2}\frac{U_0}{\nu x}\,\delta g'_{\Delta x} + g_0\,\alpha\sin\alpha\,(x - U_0\,t)\cdot\left(\alpha^2 + \overline{\alpha_i^2}\frac{U_0}{\nu x}\right) = 0,$$

with

$$g'_{x_{ext}} = -\,g_0\,\alpha\sin\alpha\,(x - U_0\,t).$$

Step by step, we then obtain

$$\delta g''_{\Delta x^2} - \overline{\alpha_i^2}\frac{U_0}{\nu x_0}\delta g'_{\Delta x} + g_0\Big[\alpha \sin\alpha(x_0 - U_0 t)\cdot\Big(\alpha^2 + \overline{\alpha_i^2}\frac{U_0}{\nu x_0}\Big)$$
$$- \Delta x\Big\{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}\cdot\frac{\alpha}{x_0}\sin\alpha(x_0 - U_0 t) - \alpha^2\cos\alpha(x_0 - U_0 t)\cdot\Big(\alpha^2 + \overline{\alpha_i^2}\frac{U_0}{\nu x_0}\Big)\Big\}\Big] = 0,$$

whence

$$\delta g'_x = \sqrt{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}}\left(\delta g_1 e^{\sqrt{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}}\Delta x} - \delta g_2 e^{-\sqrt{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}}\Delta x}\right) + b' + 2c'\Delta x,$$

where b' and c' result from the general equation

$$-(b' + 2c'\Delta x)\overline{\alpha_i^2}\frac{U_0}{\nu x_0} + g_0\Big[\alpha\sin\alpha(x_0 - U_0 t)\cdot\Big(\alpha^2 + \overline{\alpha_i^2}\frac{U_0}{\nu x_0}\Big)$$
$$+ \Delta x\Big\{\frac{\alpha}{x_0}\cdot\frac{\overline{\alpha_i^2}U_0}{\nu x_0}\sin\alpha(x_0 - U_0 t) - \alpha^2\cos\alpha(x_0 - U_0 t)\cdot\Big(\alpha^2 + \overline{\alpha_i^2}\frac{U_0}{\nu x_0}\Big)\Big\}\Big] = 0,$$

so that

$$b' = g_0\left[\alpha\sin\alpha(x_0 - U_0 t)\cdot\frac{\alpha^2 + \overline{\alpha_i^2}\frac{U_0}{\nu x_0}}{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}}\right],$$

$$2c' = g_0\left[\frac{\alpha}{x_0}\sin\alpha(x_0 - U_0 t) - \alpha^2\cdot\cos\alpha(x_0 - U_0 t)\cdot\frac{\alpha^2 + \overline{\alpha_i^2}\frac{U_0}{\nu x_0}}{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}}\right].$$

For δf_x and $\delta g'_x$, we thus obtain the following expressions: /100

$$\delta f'_x = \beta(\delta f_1 - \delta f_2) + b + \Delta x\{\beta^2(\delta f_1 + \delta f_2) + 2c\} \quad \left(\text{where } \beta = \sqrt{\overline{\alpha_i^2}\frac{U_0}{\nu x_0} - k^2}\right)$$
$$\delta g'_x = \sqrt{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}}(\delta g_1 - \delta g_2) + b' + \Delta x\Big\{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}(\delta g_1 + \delta g_2) + 2c'\Big\}$$

and, for $f'_{x_{ext}}$ and $g'_{x_{ext}}$,

$$f'_{x_{ext}} = -f_0[\alpha\sin\alpha(x_0 - U_0 t) + \Delta x\,\alpha^2\cos\alpha(x_0 - U_0 t)\ldots]$$
$$g'_{x_{ext}} = -g_0[\alpha\sin\alpha(x_0 - U_0 t) + \Delta x\,\alpha^2\cos\alpha(x_0 - U_0 t)\ldots].$$

Finally, $\varphi = \varphi_1 e^{ky} + \varphi_2 e^{-ky}$, as demonstrated above.

28.4 Boundary Conditions

In $u' = \varphi'_y f$, $v' = -\varphi f'_x - g'_x$, to make the boundary conditions appear, let us enter the following:

$$u'(y=0)=0, \quad v'(y=0)=0, \quad v'(y=\delta)=v'_{ext}(\delta), \text{ finally } u'(y=\delta)=u'_{ext}(\delta)$$

$$u'(y=0)=0 \qquad \text{yields directly} \qquad k(\varphi_1-\varphi_2)=0, \text{ whence } \varphi_1=\varphi_2.$$

It is possible to take $\frac{1}{2}$ as the common value (since φ forms a product with f such that $f_1\frac{1}{2}$ will replace $f_1\varphi_1$ without reducing the generality of the reasoning).

Then,

$$\varphi=\cosh ky, \qquad \varphi'_y=k\sinh ky.$$

$$\begin{aligned} v'(y) = &-\cosh ky\,[b+\beta(\delta f_1-\delta f_2)-f_0\,\alpha\sin\alpha(x_0-U_0t) \\ &+\Delta x\{2c+\beta^2(\delta f_1+\delta f_2)-f_0\,\alpha^2\cos\alpha(x_0-U_0t)\}] \\ &-\left[\sqrt{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}}(\delta g_1-\delta g_2)+b'-g_0\,\alpha\sin\alpha(x_0-U_0t)\right. \\ &\left.+\Delta x\left\{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}(\delta g_1+\delta g_2)+2c'-g_0\,\alpha^2\cos\alpha(x_0-U_0t)\right\}\right]. \end{aligned}$$

Here, $v' \equiv 0$ yields

$$\begin{aligned} &-[b+\beta(\delta f_1-\delta f_2)-f_0\,\alpha\sin\alpha(x_0-U_0t)] \\ &\quad-\left[\sqrt{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}}(\delta g_1-\delta g_2)+b'-g_0\,\alpha\sin\alpha(x_0-U_0t)\right]=0, \\ &-[2c+\beta^2(\delta f_1+\delta f_2)-f_0\,\alpha^2\cos\alpha(x_0-U_0t)] \\ &\quad-\left[\overline{\alpha_i^2}\frac{U_0}{\nu x_0}(\delta g_1+\delta g_2)+2c'-g_0\,\alpha^2\cos\alpha(x_0-U_0t)\right]=0. \end{aligned}$$

$$v'(\delta)=v'_{ext}(\delta)=(g_0+f_0)\,[\alpha\sin\alpha(x_0-U_0t)+\Delta x\,\alpha^2\cos\alpha(x_0-U_0t)\ldots]$$

will also furnish /101

$$\begin{aligned} &-\cosh k\delta\,[b+\beta(\delta f_1-\delta f_2)]-\left[\sqrt{\overline{\alpha_i^2}\frac{U_0}{\nu x_0}}(\delta g_1-\delta g_2)+b'\right] \\ &=f_0(1-\cosh k\delta)\,\alpha\sin\alpha(x_0-U_0t), \\ &-\cosh k\delta\,[2c+\beta^2(\delta f+\delta f_2)-\left[\overline{\alpha_i^2}\frac{U_0}{\nu x_0}(\delta g_1+\delta g_2)+2c'\right] \\ &=f_0(1-\cosh k\delta)\,\alpha^2\cos\alpha(x_0-U_0t). \end{aligned}$$

By difference, we obtain

$$
\begin{aligned}
&-(\cosh k\delta - 1)\,[b + \beta(\delta f_1 - \delta f_2) - f_0\,\alpha \sin\alpha(x_0 - U_0 t)] \\
&= (f_0 + g_0)\,\alpha \sin\alpha(x_0 - U_0 t), \\
&-(\cosh k\delta - 1)\,[2c + \beta^2(\delta f_1 + \delta f_2) - f_0\,\alpha^2 \cos\alpha(x_0 - U_0 t)] \\
&= (f_0 + g_0)\,\alpha^2 \cos\alpha(x_0 - U_0 t),
\end{aligned}
$$

whence:

$$
b + \beta(\delta f_1 - \delta f_2) \quad \text{and} \quad 2c + \beta^2(\delta f_1 + \delta f_2), \qquad \text{i.e.,} \qquad \delta f_1 \text{ and } \delta f_2.
$$

Similarly, it is easy to eliminate δf_1 for determining δg_1 and δg_2 such that

$$
\begin{aligned}
&b' + \sqrt{\overline{\alpha_i^2}\,\frac{U_0}{\nu x_0}}\,(\delta g_1 - \delta g_2) - g_0\,\alpha \sin\alpha(x_0 - U_0 t) \\
&= -[b + \beta(\delta f_1 - \delta f_2) - f_0\,\alpha \sin\alpha(x_0 - U_0 t)] \\
&= \frac{1}{\cosh k\delta - 1}\,[f_0 + g_0]\,\alpha \sin\alpha(x_0 - U_0 t), \\
&2c' + \overline{\alpha_i^2}\,\frac{U_0}{\nu x_0}\,(\delta g_1 + \delta g_2) - g_0\,\alpha^2 \cos\alpha(x_0 - U_0 t) \\
&= \frac{1}{\cosh k\delta - 1}\,(f_0 + g_0)\,\alpha^2 \cos\alpha(x_0 - U_0 t).
\end{aligned}
$$

We then obtain

$$
\begin{aligned}
v'(y) &= \frac{f_0 + g_0}{\cosh k\delta - 1}\,(\cosh ky - 1)\,[\alpha \sin\alpha(x_0 - U_0 t) + \Delta x\,\alpha^2 \cos\alpha(x_0 - U_0 t) \ldots] \\
&= \frac{f_0 + g_0}{\cosh k\delta - 1}\cdot(\cosh ky - 1)\,\alpha \sin\alpha(x - U_0 t).
\end{aligned}
$$

So far as $u'(y)$ is concerned, its expression reads

$$
\begin{aligned}
u'(y) &= k \sinh ky\,[f(x_0 - U_0 t) + \Delta x\,f'_x(x_0 - U_0 t) \ldots] \\
&= k \sinh ky \cdot f(x - U_0 t),
\end{aligned}
$$

which can be written directly since $\delta_x' \cong 0$ relative to f_x' and g_x': /102

$$
u'(y) = \frac{f_0 + g_0}{\cosh k\delta - 1}\,[\cos\alpha(x - U_0 t) + \text{const}]\,k \sinh ky.
$$

In fact, according to the expression of $v'(y) = -[\cosh ky \cdot f_x' + g_x']$, we have

$$
f'_x(x - U_0 t) = -\frac{f_0 + g_0}{\cosh k\delta - 1}\,\alpha \sin\alpha(x - U_0 t)
$$

and

$$f(x - U_0 t) = \frac{f_0 + g_0}{\cosh k\delta - 1} \cos\alpha (x - U_0 t) + \text{const.}$$

The constant incorporates the integration constant δf_3 of the integration of

$$\delta f = \int \delta f'_x \cdot dx + \delta f_3.$$

This will make it possible to satisfy any initial condition with respect to x concerning $u'(y)$.

28.5 Determination of k

The constant k is still indeterminate.

Let us note that the expressions of φ and f, relative to the domain interior to the boundary layer, are as follows:

$$\varphi = \cosh ky,$$

with

$$f'_x + \delta f'_x = -\frac{f_0 + g_0}{\cosh k\delta - 1} \alpha \sin\alpha (x - U_0 t) = \frac{\frac{g_0}{f_0} + 1}{\cosh k\delta - 1} \cdot f'_{x_{ext}}$$

and

$$f'''_{x^3} + \delta f'''_{x^3} \cong \frac{f_0 + g_0}{\cosh k\delta - 1} \alpha^3 \sin\alpha (x - U_0 t) = -\frac{\frac{g_0}{f_0} + 1}{\cosh k\delta - 1} \cdot \alpha^2 f'_{x_{ext}}.$$

However, we were forced to choose a value δ for the thickness of the laminar boundary layer $\left(\delta \cong 5.5 \sqrt{\frac{\nu x}{U_0}}\right)$, fixing an extent limited to the application domain of the Blasius theory (see Sect.28.1 above). This choice had been such that, for $y \geq \delta$, the value of $\frac{U}{U_0}$ calculated by this theory just about reaches unity.

This means that, in $y = \delta$, a connection must exist between the interior and the exterior solution and that the connectivity conditions must be satisfied irrespective of the selected $y > \delta$, since any other value $\delta' > \delta$ could have been assigned to δ (obviously, within the frame of the agreed approximations), specifically in so far as the rotations are concerned.

Let us return to the fundamental condition, referring to the exterior domain. This condition was written in the form (see Sect.28.2) /103

$$[\varphi''_{y^2} f'_x + \varphi f'''_{x^3}]_{ext} = 0$$

(representing the law of conservation of rotations) and was satisfied for

$$\left[\frac{\varphi_{y^2}}{\varphi}\right]_{ext} = -\left[\frac{f'''_{x^3}}{f'_x}\right]_{ext} = \alpha^2.$$

In the interior domain, according to what has been demonstrated above, we have

$$f'_x = f'_{x_{ext}} + \delta f'_x = + f'_{x_{ext}} \cdot \frac{\frac{g_0}{f_0} + 1}{\cosh k\delta - 1} \qquad \text{and} \qquad \varphi = \cosh ky.$$

Let us form

$$\frac{\varphi''_{y^2}}{\varphi} \qquad \text{and} \qquad \frac{f'''_{x^3}}{f'_x}.$$

These ratios* will satisfy the above connectivity condition if and only if $k^2 = \alpha^2$.

This fixes the value of the constant k.

28.6 Expressions of the Interior Perturbation Components

Next, the values to be retained for $u'(y)$ and $v'(y)$ can be written down:

$$u'(y, x) = \alpha \sinh \alpha y \left[\frac{f_0 + g_0}{\cosh \alpha\delta - 1} \cos \alpha (x - U_0 t) + \text{const}\right],$$

$$v'(y, x) = (f_0 + g_0) \frac{\cosh \alpha y - 1}{\cosh \alpha \delta - 1} \alpha \sin \alpha (x - U_0 t).$$

Thus, along the border δ at the interior of the boundary layer, we have

$$u'_{int}(\delta) = \frac{\alpha \sinh \alpha \delta}{\cosh \alpha \delta - 1} [(f_0 + g_0) \cos \alpha (x - U_0 t) + \text{const}],$$

$$v'_{int}(\delta) = (f_0 + g_0) \alpha \sin \alpha (x - U_0 t).$$

Consequently, an external perturbation in $\psi_{ext} = F_0 e^{-\alpha(\delta - y)} \cos \alpha(x - U_0 t)$, which tends to zero as $y \to \infty$, has the following (exterior) velocity components in δ:

$$v'_{ext}(\delta) = -F_0 \alpha \sin \alpha (x - U_0 t),$$

$$u'_{ext}(\delta) = -\alpha F_0 \cos \alpha (x - U_0 t).$$

* Here, δ'_x has always been assumed as negligible compared with f'_x, f'''_{x^3}.

Connectedness of the exterior and interior components in u′ along δ, will be ensured if /104

$$-\alpha F_0 \cos \alpha (x - U_0 t) \cong \alpha \cdot (f_0 + g_0) \frac{\sinh \alpha \delta}{\cosh \alpha \delta - 1} \cos \alpha (x - U_0 t).$$

It follows then

$$F_0 \cong -(f_0 + g_0) \frac{\sinh \alpha \delta}{\cosh \alpha \delta - 1},$$

whence

$$v'_{ext}(\delta) = F_0 \alpha \sin \alpha (x - U_0 t) = (f_0 + g_0) \alpha \sin \alpha (x - U_0 t) \frac{\sinh \alpha \delta}{\cosh \alpha \delta - 1},$$

whereas

$$v'_{int}(\delta) = (f_0 + g_0) \alpha \sin \alpha (x - U_0 t).$$

So as to have a connectivity exist between the velocity components v′ at the exterior and at the interior of the layer along δ, it is necessary to supplement the external perturbation by a second perturbation in $\Delta v'_{ext}$:

$$\Delta v'_{ext} = v'_0 \sin \alpha (x - U_0 t),$$

such that

$$v'_0 + \frac{\sinh \alpha \delta}{\cosh \alpha \delta - 1} (f_0 + g_0) \alpha = (f_0 + g_0) \alpha,$$

whence

$$v'_0 = -\alpha (f_0 + g_0) \frac{1 + \sinh \alpha \delta - \cosh \alpha \delta}{\cosh \alpha \delta - 1}.$$

It is necessary that no component in u′ be attached to this component in v'_0 (meaning that it should be of the type investigated in Sect.26).

This also means that the laminar boundary layer responds to such a harmonic perturbation in $v'_{0_{ext}}$ (only) by an internal perturbation of the above-defined type which, again at the interior, produces the occurrence of a complementary external perturbation (in v'_{ext} and u'_{ext}) which rapidly decays with increasing y beyond δ.

29. Limit to the Extension of the Laminar Domain

Let us return to the Navier-Stokes equations with perturbations u′, v′ calculated above for the laminar boundary layer. These equations are as follows:

$$\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{\partial U}{\partial y} + \frac{1}{\rho} \cdot \frac{\partial p'}{\partial x} = \nu \cdot \Delta^2 u',$$

$$\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} + \frac{1}{\rho} \cdot \frac{\partial p'}{\partial y} = \nu \cdot \Delta^{2'} v,$$

where

/105

$$U = U_0\left[1 - \sum_i a_i e^{-\alpha_i\sqrt{\frac{U_0}{\nu x}}y}\right], \quad u' = (e^{\alpha y} - e^{-\alpha y})\,\alpha\, f.$$

Finally,

$$v' = -(e^{+\alpha y} + e^{-\alpha y})\, f'_x - g'_x,$$

where f and g satisfy the above-mentioned general conditions.

Thus,

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = \varphi'_y f'_t + U_0\left\{1 - \sum_i a_i e^{-\alpha_i y\sqrt{\frac{U_0}{\nu x}}}\right\}\varphi'_y f'_x$$

$$- U_0 \sum_i a_i \alpha_i \sqrt{\frac{U_0}{\nu x}}(\varphi f'_x + g'_x)$$

$$= \varphi'_y (f'_t + U_0 f'_x) - U_0 \sum_i a_i \cdot e^{-\alpha_i\sqrt{\frac{U_0}{\nu x}}y}\left[\varphi'_y f'_x + \alpha_i \sqrt{\frac{U_0}{\nu x}}(\varphi f'_x + g'_x x)\right]$$

$$= - U_0 \sum_i a_i e^{-\alpha_i\sqrt{\frac{U_0}{\nu x}}y}\left[\alpha (e^{\alpha y} - e^{-\alpha y}) f'_x + \alpha_i \sqrt{\frac{U_0}{\nu x}}\{(e^{\alpha y} + e^{-\alpha y}) f'_x + g'_x\}\right].$$

Similarly,

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -(\varphi f''_{xt} + g''_{x^2 t}) - U_0\left\{1 - \sum_i a_i e^{-\alpha_i\sqrt{\frac{U_0}{\nu x}}y}\right\}(\varphi f''_{x^2} + g''_{x^2})$$

$$= -\varphi (f''_{xt} + U_0 f''_{x^2}) - (g''_{xt} + U_0 g''_{x^2}) + U_0 \sum_i a_i e^{-\alpha_i\sqrt{\frac{U_0}{\nu x}}y}(\varphi f''_{x^2} + g''_{x^2})$$

$$= U_0 \sum_i a_i e^{-\alpha_i\sqrt{\frac{U_0}{\nu x}}y}[(e^{\alpha y} + e^{-\alpha y}) f''_{x^2} + g''_{x^2}].$$

These forms cause the occurrence of terms in $e^{-\{\alpha+\alpha_1\sqrt{\frac{U_0}{\nu x}}\}y} \pm$ $\pm\, e^{\{\alpha-\alpha_1\sqrt{\frac{U_0}{\nu x}}\}y}$.

When y increases from 0 to δ, the first exponential will decrease (very rapidly), while the second exponential which decreases in a similar manner at

$\alpha < \alpha_i \sqrt{\frac{U_0}{\nu x}}$ will increase rapidly at $\alpha > \sqrt{\frac{U_0}{\nu x}} \cdot \alpha_i$, which will be again the same for $\frac{\partial p'}{\partial x}$ and $\frac{\partial p'}{\partial y}$.

29.1 Consequences /106

Let x_c be the value of x which cancels $\alpha - \alpha_i \sqrt{\frac{U_0}{\nu x}}$ such that

$$\alpha^2 = \alpha_i^2 \frac{U_0}{\nu x_c}, \qquad x_c = \frac{\alpha_i^2}{\alpha^2} \cdot \frac{U_0}{\nu},$$

where $\alpha_i = \alpha_i(1 - \epsilon)$, in principle, is the smallest of the coefficients α_i taken into consideration. In fact, $\alpha_i \cong \alpha_i$.

Thus, as soon as $\frac{x - x_c}{x_c}$ is positive and not extremely small, substantial (and even rapidly increasing) pressure gradients will appear in the thickness of the boundary layer.

This means that the radii of curvature of the particle trajectories will no longer be large so that the Blasius hypothesis no longer is applicable. Thus, the Blasius solution cannot extend beyond $x = x_c$, and only the second solution of the Navier-Stokes equations studied in Part I of this paper under the designation of "stationary turbulent" solution will remain valid. For this, no hypothesis has been established with respect to the pressure gradients or the trajectory curvatures*.

Thus, it is necessary that at x_c - or at least in its immediate vicinity - the laminar state stops existing and that a second state, connected with it, takes its place.

It will be noted that this change of state is correlated with the existence of an external nonstationary perturbation which induces a response of the laminar layer of the same pulsation α; its combination with the U''_{y^2} law of the Blasius field leads to a divergence of the pressure gradients (with respect to y), starting from a well-defined critical segment x_c which depends directly on the pulsation of the external perturbation (and thus on its wavelength or on its frequency).

Later in the text (in Sect.34.1), we will determine the response of the "stationary turbulent" boundary layer to the existence of a harmonic external perturbation. The calculation is simpler than - but similar to - that carried out for the laminar boundary layer; no source for divergence of the pressure

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* The two Navier-Stokes equations were taken into consideration, between which the pressure is eliminated; thus, no hypothesis on these equations is formulated, and also no hypothesis on the smallness of the trajectory curvature radii.

gradients and for a limitation of extension exist here.

No matter how this might be, the following critical Reynolds number corresponds to x_c:

$$\mathcal{R}_c = \frac{U_0 x_c}{\nu} = \left(\frac{\overline{\alpha_i}}{\alpha} \cdot \frac{U_0}{\nu}\right)^2$$

which is higher the smaller α and ν and the larger U_0. Here, $U_0\alpha$ characterizes the pulsation of the external perturbation, whose frequency and wavelength are as follows: /107

$$\lambda = \frac{U_0}{N} = \frac{2\pi}{\alpha}, \qquad N = \frac{U_0 \alpha}{2\pi}.$$

This yields

$$\mathcal{R}_c = \frac{\overline{\alpha_i^2}}{4\pi^2} \cdot \frac{\lambda^2 U_0^2}{\nu^2},$$

where $\Re_\lambda = \dfrac{\lambda U_0}{\nu}$ thus is the parameter determining $\Re_c$.

Here, $\overline{\alpha_i}$ is the expansion factor representing the Blasius velocity profile:

$$U = U_0 [1 - e^{-0.375} \{ 73^5. - 72^5. \cosh 0.037^5 \cdot \eta \}]$$

with

$$\eta = y\sqrt{\frac{U_0}{\nu x}}, \qquad \overline{\alpha_i} = 0.37^5.$$

Let us define the orders of magnitude of λ (and N) to which various possible values of $\Re_c$ correspond, when $\nu = 14.4 \times 10^{-6}$ MKS:

	$\mathcal{R}_c$	$0.3 \cdot 10^6$	$1 \cdot 10^6$	$2 \cdot 10^6$
For $U_0 = 29$ m/sec	λ (m)	$4.5 \cdot 10^{-3}$	$8.2 \cdot 10^{-3}$	$11.2 \cdot 10^{-3}$
	N (cps)	$6.6 \cdot 10^3$	$3.5 \cdot 10^3$	$2.5 \cdot 10^3$
For $U_0 = 87$ m/sec	λ (m)	$1.5 \cdot 10^{-3}$	$2.75 \cdot 10^{-3}$	$3.9 \cdot 10^{-3}$
	N (cps)	$58 \cdot 10^3$	$30 \cdot 10^3$	$21 \cdot 10^3$.

The wavelengths are expressed in millimeters, and the frequencies in kilocycles or tens of kilocycles.

CHAPTER V /108

PROPAGATION OF A PERTURBATION IN u′, INTERIOR TO THE BOUNDARY LAYER

30. Perturbation u′ Interior to the Boundary Layer

Since the fundamental velocity field is of the Blasius type, such that

$$U \cong U_0\left[1-\sum_i a_i e^{-\alpha_i y\sqrt{\frac{U_0}{\nu x}}}\right], \qquad U''_{y^2} \cong -U_0\sum_i a_i\,\alpha_i^2\frac{U_0}{\nu x}e^{-\alpha_i y\sqrt{\frac{U_0}{\nu x}}},$$

let us here consider a nonstationary perturbation stream function u′, v′:

$$\psi = \varphi(y)\cdot f(x, t) + g(x, t).$$

The fundamental condition is written in the form (see Sect.27):

$$\varphi''_{y^2}(f'_x + U f'_x) + \varphi(f'''_{x^2t} + U f'''_{x^3} - U''_{y^2} f'_x) + (g'''_{x^2t} + U g'''_{x^3} - U''_{y^2} g'_x)$$
$$= \nu\,[f\,\varphi''''_{y^4} + 2 f''_{x^2}\varphi''_{y^2} + \varphi f''''_{x^4} + g''''_{x^4}],$$

and leads here to

$$U_0\left(1-\sum_i a_i e^{-\alpha_i y\sqrt{\frac{U_0}{\nu x}}}\right)[\varphi''_{y^2} f'_x + \varphi f'''_{x^3} + g'''_{x^3}]$$
$$+ U_0\sum_i a_i\cdot\alpha_i^2\frac{U_0}{\nu x}e^{-\alpha_i y\sqrt{\frac{U_0}{\nu x}}}(\varphi f'_x + g'_x) + \varphi''_{y^2} f'_t + \varphi f'''_{x^2t} + g'''_{x^2t}$$
$$= \nu\,[f\,\varphi''''_{y^4} + 2\varphi''_{y^2} f''_{x^2} + \varphi f''''_{x^4} + g''''_{x^4}].$$

This condition is decomposed into so-called "time" relations

$$\varphi''_{y^2}(f'_t + U_0 f'_x) + \varphi(f'''_{xt^2} + U_0 f'''_{x^3}) + g'''_{x^2t} + U_0 g'''_{x^3}$$
$$= \nu\,[f\,\varphi''''_{y^4} + 2 f''_{x^2}\varphi''_{y^2} + \varphi f''''_{x^4} + g''''_{x^4}]$$

and into so-called "space" relations /109

$$a_i e^{-\alpha_i y\sqrt{\frac{U_0}{\nu x}}}\left[\varphi''_{y^2} f'_x + \varphi\left(f'''_{x^3} - \alpha_i^2\frac{U_0}{\nu x}f'_x\right) + g'''_{x^3} - \alpha_i^2\frac{U_0}{\nu x}g'_x\right] = 0.$$

We will then return to the three-term expansion, used previously, such that

$$\frac{\Phi''_{\eta^2}}{\Phi} = \mathrm{const}\,\bar{\alpha}_i^2 \quad (\text{where } \alpha_3 = \bar{\alpha}_i, \qquad \alpha_{1,2} = \bar{\alpha}_i(1 \mp \varepsilon), \qquad \varepsilon \text{ small}),$$

so that, as above, we will have to solve the following system of equations (since α_1 can no longer be distinguished from $\overline{\alpha_1}$, we do not wish - in what follows - to complicate the writing by using the vinculum $\alpha_1 = \overline{\alpha_1}$).

The above conditions must be resolved into conditions connected with y and into conditions independent of y. For the time relations, we have

$$\frac{\partial}{\partial t}[\varphi''_{y^2} f + \varphi f''_{x^2}] + U_0 [\varphi''_{y^2} f'_x + \varphi f'''_{x^3}] = \nu [f \varphi''''_{y^4} + 2 \varphi''_{y^2} f''_{x^2} + \varphi f''''_{x^4}] \tag{A}$$

and

$$\frac{\partial}{\partial t} g''_{x^2} + U_0 g'''_{x^3} = \nu g''''_{x^4}. \tag{A'}$$

For the space relations, we obtain

$$\varphi''_{y^2} f'_x + \varphi\left(f'''_{x^3} - \alpha_i^2 \frac{U_0}{\nu x} f'_x\right) = 0 \tag{B}$$

and

$$g'''_{x^3} - \alpha_i^2 \frac{U_0}{\nu x} g'_x = 0. \tag{B'}$$

Since φ is a function of y alone, eq.(B) of the second order in $\varphi(y)$ will furnish

$$\frac{\varphi''_{y^2}}{\varphi} = -\frac{f'''_{x^3} - \alpha_i^2 \frac{U_0}{\nu x} f'_x}{f'_x} = k^2,$$

where k is a constant with respect to x as well as to y; for f, we then obtain the following equation of the second order in f'_x:

$$f'''_{x^3} + f'_x\left((k^2 - \alpha_i^2 \frac{U_0}{\nu x}\right) = 0.$$

31. Study of First Approximation /110

Let us investigate f about an arbitrary but particularized value $x = x_0$, $t = t_0$, at weak variations Δx, Δt.

In this step-by-step procedure, in accordance with the formation of k, the expression $\beta^2 = \alpha_1^2 \frac{U_0}{\nu x_0} - k^2$ will be considered as constant in first approximation. The solution f'_x will have the form

$$f'_x = \beta\,(f_1 e^{\beta\Delta x} - f_2 e^{-\beta\Delta x})$$

and

$$f = f_1 e^{\beta \Delta x} + f_2 e^{-\beta \Delta x} + f_3,$$

where f_1, f_2, f_3 are functions of x_0, t_0, and Δt.

In the same manner, we obtain

$$g'_x = \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \left(g_1 \, e^{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Delta x} - g_2 \, e^{-\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Delta x} \right),$$

$$g = g_1 \, e^{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Delta x} + g_2 \, e^{-\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Delta x} + g_3.$$

Let us go back to the time equation in g. For the terms in g_1, we obtain

$$\alpha_i^2 \frac{U_0}{\nu x_0} \left[g'_{1t} + U_0 \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \, g_1 - \nu \, g_1 \, \alpha_i^2 \frac{U_0}{\nu x_0} \right] = 0,$$

i.e.,

$$g_1 = g_{1_0} \, e^{-U_0 \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \left(1 - \frac{\nu}{U_0} \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}\right) \Delta t} \cong g_{1_0} \, e^{-U_0 \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Delta t}$$

Similarly,

$$g_2 \cong g_{2_0} \, e^{U_0 \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Delta t}$$

and, finally,

$$g_3 = g_{3_0}.$$

Here, g_{1_0}, g_{2_0}, g_{3_0} are constants so that we have

$$g \cong g_{1_0} \, e^{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} (\Delta x - U_0 \Delta t)} + g_{2_0} \, e^{-\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} (\Delta x - U_0 \Delta t)} + g_{3_0}.$$

Simultaneously,

$$\varphi = \varphi_1 e^{ky} + \varphi_2 e^{-ky}.$$

Let us enter these forms in the time equation (A). Since this is written as /111

$$\frac{\partial}{\partial t} [\varphi''_{y^2} f + \varphi f''_{x^2}] + U_0 \left[(\varphi''_{y^2} f'_x + \varphi f'''_{x^3}) - \frac{\nu}{U_0} (f \varphi''''_{y^4} + 2 \varphi''_{y^2} f''_{x^2} + \varphi f''''_{x^4}) \right] = 0$$

the following relation will be attached to each exponential $e^{\pm ky}$:

$$\frac{\partial}{\partial t} k^2 f_3 - \frac{\nu}{U_0} k^4 f_3 + \frac{\partial}{\partial t}[f_1(\beta^2 + k^2) e^{\beta\Delta x} + f_2(\beta^2 + k^2) e^{-\beta\Delta x}]$$

$$+ U_0\beta\left[f_1(k^2+\beta^2) e^{\beta\Delta x} - f_2(k^2+\beta^2) e^{-\beta\Delta x} - \frac{\nu}{U_0}\{f_1(k^2+\beta^2)^2 e^{\beta\Delta x} + f_2(k^2+\beta^2)^2 e^{-\beta\Delta x}\}\right] = 0,$$

i.e.,

$$f'_{1_t} + f_1 U_0 \beta\left[1 - \frac{\nu}{U_0}\cdot\frac{k^2+\beta^2}{\beta}\right] = 0,$$

so that

$$f_1 = f_{1_0} e^{-U_0\left\{\beta - \frac{\nu}{U_0}(\beta^2+k^2)\right\}\Delta t} \cong f_{1_0} e^{-U_0\beta\Delta t};$$

$$f'_{2_t} - f_2 U_0 \beta\left[1 + \frac{\nu}{U_0}\cdot\frac{k^2+\beta^2}{\beta}\right] = 0,$$

so that

$$f_2 = f_{2_0} e^{U_0\left\{\beta + \frac{\nu}{U_0}(\beta^2+k^2)\right\}\Delta t} \cong f_{2_0} e^{U_0\beta\Delta t};$$

$$f'_{3_t} - \frac{\nu}{U_0} k^2 f_3 = 0,$$

so that

$$f_3 = f_{3_0} e^{\frac{\nu}{U_0}k^2\Delta t} \cong f_{3_0},$$

where f_{1_0}, f_{2_0}, f_{3_0} are constants with respect to Δx, Δt (but dependent on the step-by-step origin conditions, i.e., on x_0, t_0).

Consequently, the general expression of f will be, with

$$\beta = \sqrt{\alpha_i^2\frac{U_0}{\nu x_0} - k^2}$$

$$f = \left[f_{1_0} e^{\sqrt{\alpha_i^2\frac{U_0}{\nu x_0} - k^2}(\Delta x - U_0\Delta t)} + f_{2_0} e^{-\sqrt{\alpha_i^2\frac{U_0}{\nu x_0} - k^2}(\Delta x - U_0\Delta t)}\right] e^{\alpha_i^2\frac{U_0}{x_0}\Delta t} + f_{3_0} e^{\frac{\nu}{U_0}k^2\Delta t}$$

$$= f_{1_0} e^{\sqrt{\alpha_i^2\frac{U_0}{\nu x_0} - k^2}\left\{\Delta x - U_0\left(1 - \frac{\alpha_i^2}{x_0}\frac{1}{\sqrt{\alpha_i^2\frac{U_0}{\nu x_0} - k^2}}\right)\Delta t\right\}}$$

$$+ f_{2_0} e^{-\sqrt{\alpha_i^2\frac{U_0}{\nu x_0} - k^2}\left\{\Delta x - U_0\left(1 + \frac{\alpha_i^2}{x_0}\frac{1}{\sqrt{\alpha_i^2\frac{U_0}{\nu x_0} - k^2}}\right)\Delta t\right\}} + f_{3_0} e^{\frac{\nu}{U_0}k^2\Delta t}.$$

31.1 Boundary Conditions

Here, the boundary conditions with respect to y will be those referring to the wall $y = 0$, for which $v'(0) = 0$ and, in general, $u'(0) = 0$.

In $y = \delta$, the exterior flow may be affected indirectly by perturbations inside the boundary layer. It would be sufficient to establish simply that a perturbation exists in the exterior flow whose boundary conditions along the boundary δ ensure connectivity with the internal perturbation. At $y = \infty$, the external perturbation would vanish. On the basis of the solutions of the family of those studied in Sections 24, 25, and 26, it is easy to obtain the means for defining such a perturbation. All this merely is to prove that we can disregard the conditions at the boundary of the internal perturbation.

So as to have $u(0)$ be zero, let us set $\varphi_1 = \varphi_2 = \frac{1}{2}$ (since φ_1 and φ_2 always form a product with f, the generality of the solution is not impaired).

Hence,

$$u'(y) = fk \sinh ky, \qquad v'(y) = -f'_x \cosh ky - g'_x.$$

Here, $v'(0) = 0$ leads to $f'_x + g'_x = 0$.

Let us also put $\Delta X = \Delta x - U_0 \Delta t$ and let us expand the exponentials

$$e^{\pm\sqrt{\alpha_i^2 \frac{U_0}{\nu x} - k^2}(\Delta x - U_0 \Delta t)} \cong 1 \pm \sqrt{\alpha_i^2 \frac{U_0}{\nu x} - k^2} \cdot \Delta X.$$

If $\beta = \sqrt{\alpha_i^2 \frac{U_0}{\nu x} k^2}$ is real, all coefficients f_{1_0}, f_{2_0}, g_{1_0}, g_{2_0} will also be real. However, if β is imaginary, then $\beta = \mathring{i}\bar{\beta}$, $f_{1_0} = f^*_{1_0} + \mathring{i} f^{**}_{1_0}$. This is the same for f_{2_0} (and, by homogeneity, $f_{3_0} = f^*_{3_0}$).

In expanded form, the condition $v'(0) = 0$ reads as follows:

$$f'_x + g'_x = \left[\left\{ (f_{1_0} - f_{2_0})\beta + (g_{1_0} - g_{2_0}) \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \right\} + \Delta x \left\{ (f_{1_0} + f_{2_0})\beta^2 + (g_{1_0} + g_{2_0}) \alpha_i^2 \frac{U_0}{\nu x_0} \right\} \right] = 0.$$

If β is real, it is necessary that

$$f_{1_0} = f_{2_0}, \qquad g_{1_0} = g_{2_0}, \qquad g_{1_0} = -f_{1_0} \frac{\alpha_i^2 \frac{U_0}{\nu x_0} - k^2}{\alpha_i^2 \frac{U_0}{\nu x_0}}.$$

If $\beta = \mathring{i}\bar{\beta}$, with $\bar{\beta}$ being real,

$$f'_x + g'_x = \Big[\Big\{ \bar\beta \big(\overset{\circ}{i} (f^*_{1_0} - f^*_{2_0}) - (f^{**}_{1_0} - f^{**}_{2_0}) \big) + (g_{1_0} - g_{2_0}) \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Big\}$$
$$+ \Delta X \Big\} \bar\beta^2 \big(\overset{\circ}{i} (f^{**}_{1_0} + f^{**}_{2_0}) - (f^*_{1_0} + f^*_{2_0}) \big) + (g_{1_0} + g_{2_0}) \alpha_i^2 \frac{U_0}{\nu x_0} \Big\{ \ldots \Big] = 0$$

will lead to $f^*_{1_0} = f^*_{2_0}$, $f^{**}_{1_0} = -f^{**}_{2_0}$ since f and f'_x are real; consequently, /113

$$g_{1_0} - g_{2_0} = 2 f^{**}_{1_0} \sqrt{\frac{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}{\alpha_i^2 \frac{U_0}{\nu x_0}}}, \qquad g_{1_0} + g_{2_0} = \frac{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}{\nu_i^2 \frac{U_0}{\nu x_0}} 2 f^*_{1_0}.$$

Thus, it is easy to calculate g_{1_0} and g_{2_0} as soon as $f^*_{1_0}$ and $f^{**}_{1_0}$ are known.

Then, the following expression will be obtained for f:

$$f = [2 f^*_{1_0} \cos \bar\beta (\Delta x - U_0 \Delta t) - 2 f^{**}_{1_0} \sin \bar\beta (\Delta x - U_0 \Delta t)]\, e^{\alpha_i^2 \frac{U_0}{\nu x_0} \Delta t} + f^*_{3_0} e^{\frac{\nu}{U_0} k^2 \Delta t},$$

where $\bar\beta = \sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}$ is a function of x_0.

Let us put $\tan \Phi = \frac{f^*_{1_0}}{f^{**}_{1_0}}$:

$$f = 2 \frac{f^{**}_{1_0}}{\cos \Phi} \sin \{ \bar\beta (\Delta x - U_0 \Delta t) - \Phi \}\, e^{\alpha_i^2 \frac{U_0}{x_0} \Delta t} + f^*_{3_0} e^{\frac{\nu}{U_0} k^2 \Delta t}.$$

Since the origin of time is arbitrary, it is always possible to select this origin, for the step under consideration, such that the new time will be expressed with respect to the first time by $\Delta t' = \Delta t + \frac{\Phi}{\bar\beta U_0}$.

Let us then put

$$\bar f_{1_0} = -\frac{f^{**}_{1_0}}{\cos \Phi} e^{-\alpha_i^2 \frac{\Phi}{\bar\beta x_0}}, \qquad \bar f_{3_0} = f^*_{3_0} e^{-k^2 \cdot \frac{\nu \Phi}{U_0^2 \bar\beta}}.$$

Hence,

$$f = 2 \bar f_{1_0} \sin \bar\beta (\Delta x - U_0 \Delta t')\, e^{\alpha_i^2 \frac{U_0}{x_0} \Delta t'} + \bar f_{3_0} e^{\frac{\nu}{U_0} k^2 \Delta t'}$$
$$= \bar f_{1_0} \big(e^{\beta(\Delta x - U_0 \Delta t')} + e^{-\beta(\Delta x - U_0 \Delta t')} \big) e^{\alpha_i^2 \frac{U_0}{x_0} \Delta t'} + \bar f_{3_0} e^{\frac{\nu}{U_0} k^2 \Delta t'}.$$

Since $\beta = \overset{\circ}{i} \bar\beta$, this form is the same as that referring to the real case β, except that, to each step, a particular origin of time must correspond.

Thus, as long as a step-by-step numerical calculation is not required, we can use a contracted form of writing, even for the imaginary case β. We will have need of this in studying the second approximation.

31.2 Propagation /114

Let us now investigate the propagation conditions.

First case:

It is assumed that β is real, i.e.,

$$\alpha_l^2 \frac{U_0}{\nu x_0} - k^2 > 0 \qquad \text{or} \qquad x_0 < \frac{\alpha_l^2}{k^2} \cdot \frac{U_0}{\nu} = \bar{x}_0.$$

In that case, the coefficient f_{1_0} is connected with a perturbation whose velocity of propagation reads

$$\mathfrak{A}_1 = U_0 \left[1 - \frac{\alpha_l^2}{x_0} \cdot \frac{1}{\sqrt{\alpha_l^2 \frac{U_0}{\nu x_0} - k^2}} \right].$$

Similarly, for f_{2_0},

$$\mathfrak{A}_2 = U_0 \left[1 + \frac{\alpha_l^2}{x_0} \cdot \frac{1}{\sqrt{\alpha_l^2 \frac{U_0}{\nu x_0} - k^2}} \right].$$

Here, $\mathfrak{A}_2$ is constantly positive, while $\mathfrak{A}_1$ vanishes for

$$\sqrt{\alpha_l^2 \frac{U_0}{\nu x_0} - k^2} - \frac{\alpha_l^2}{x_0} = 0,$$

i.e.,

$$\alpha_l^2 \frac{U_0}{\nu x_0} - k^2 - \frac{\alpha_l^4}{x_0^2} = 0.$$

Since

$$\frac{\alpha_l^4}{x_0^2} = \frac{1}{x_0^2} \left(\alpha_l^2 \frac{U_0}{\nu x_c} \right)^2 \cdot \left(\frac{\nu x_c}{U_0} \right)^2,$$

where x_c is the abscissa of the critical segment defined in Section 29.1, it follows that*

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* Let us recall that $x_c = \dfrac{\alpha_1^2}{\alpha^2} \cdot \dfrac{U_0}{\nu}$ is the abscissa where the laminar Blasius solution no longer is applicable if, in the exterior flow, there exists a perturbation of pulsation αU_0.

$$\alpha^2 \cdot \frac{x_c}{x_0} - k^2 - \alpha^4 \cdot \left(\frac{x_c}{x_0}\right)^2 \cdot \left(\frac{\nu}{U_0}\right)^2 = 0$$

or

$$\frac{x_0^2}{x_c^2} - \frac{\alpha^2}{k^2} \cdot \frac{x_0}{x_c} + \frac{\alpha^4}{k^2}\left(\frac{\nu}{U_0}\right)^2 = 0$$

whose roots are x_1, x_2 such that /115

$$\frac{x_{1,2}}{x_c} = \frac{\alpha^2}{2k^2} \pm \sqrt{\left(\frac{\alpha^2}{2k^2}\right)^2 - \frac{\alpha^4}{k^2}\left(\frac{\nu}{U_0}\right)^2} \cong \frac{\alpha^2}{2k^2}\left[1 \pm \left(1 - \frac{1}{2}4\left(\frac{\nu}{U_0}\right)^2 k^2\right)\right],$$

so that

$$\frac{x_1}{x_c} = \frac{\alpha^2}{k^2} - k^2\left(\frac{\nu}{U_a}\right)^2, \qquad \frac{x_2}{x_c} = k^2\left(\frac{\nu}{U_0}\right)^2.$$

In this form, it appears that $\frac{x_1}{x_c}$ is directly adjacent to (and inferior to) $\frac{\alpha^2}{k^2}$, while $\frac{x_2}{x_c}$ is directly adjacent to zero.

$$\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0} - k^2} = \sqrt{\alpha^2 \frac{x_c}{x_0} - k^2} \text{ vanishes for } \frac{x_0}{x_c} = \frac{\bar{x}_0}{x_c} = \frac{\alpha^2}{k^2}. \text{ Thus, } \frac{x_1}{x_c} \sim \frac{\bar{x}_0}{x_c}.$$

This discussion indicates that $\mathfrak{A}_1$ is > 0 for $x_2 < x_0 < x_1$, while $\mathfrak{A}_1$ is < 0 for x exterior to $x_1 x_2$. Thus, for $0 < x < x_2$ and $x_1 < x < \bar{x}_0$, the perturbation f_1 in the real exponential will propagate upstream, whereas it will propagate downstream for $x_2 < x < x_1$ (Fig.22).

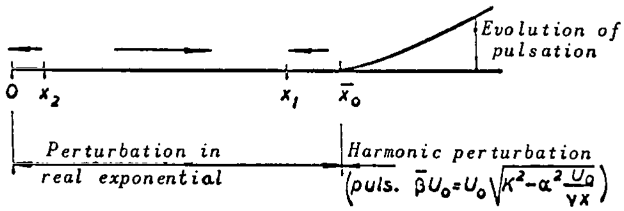

Fig.22

Second case:

Let k be real and β imaginary:

$$\alpha_i^2 \frac{U_0}{\nu x_0} - k^2 < 0, \qquad x_0 > \bar{x}_0 = \frac{\alpha_i^2}{k^2} \cdot \frac{U_0}{\nu}.$$

Then, f is written in the form

$$f = \left[f_{1_0} e^{i\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}(\Delta x - U_0 \Delta t)} + f_{2_0} e^{-i\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}(\Delta x - U_0 \Delta t)}\right] e^{\alpha_i^2 \frac{U_0}{x_0} \Delta t} + f_{3_0} e^{\frac{\nu}{U_0} k^2 \Delta t}.$$

Here, f'_x and f are necessarily real (just as φ).

As indicated above, f can always be expressed by (see Sect.31.1) /116

$$f = 2 f_{1_0} \cos\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}} (\Delta x - U_0 \Delta t) \cdot e^{+\alpha_i^2 \frac{U_0}{x_0} \Delta t} + f_{3_0} e^{\frac{\nu}{U_0} k^2 \Delta t}.$$

Thus, for $x_0 > \bar{x}_0$ (let us recall that $\bar{x}_0$ cancels the radical), the perturbation is harmonic and propagates downstream at a velocity U_0.

Upstream of $\bar{x}_0$, the perturbation is a real exponential and propagates upstream up to x_1 which it cannot overtake (since, between x_1 and x_2, this perturbation would propagate downstream). Condensation takes place in x_1 (which is very close to x_0). The exponential, bound to f_1, reduces to a constant which can only be zero since a perturbation, in the finite stationary jog, cannot be reconciled with the Navier-Stokes equations.

It should be noted here that the solution in which β is imaginary (second case) induces the appearance of a pulsation of a given value $m\alpha U_0$ (with respect to time) at a point $x = x_3$ such that

$$\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_3}} = m\alpha, \quad \text{where} \quad k^2 - m^2\alpha^2 = \alpha_i^2 \frac{U_0}{\nu x_3} = \alpha^2 \cdot \frac{x_c}{x_3},$$

so that

$$x_3 = \frac{\alpha_i^2}{k^2 - m^2\alpha^2} \cdot \frac{U_0}{\nu}.$$

Later in the text, we will have to set $x_3 = x_c$ since x_c is the critical point defined in Section 29.1 where the laminar state is unable to exist in the presence of an external perturbation of pulsation αU_0 $(m = 1)$. It follows from this that the perturbation which, in $x_3 = \bar{x}_c$, will have a pulsation $m\alpha U_0$ is characterized by a constant k such that

$$x_c = \frac{\alpha_i^2}{\alpha^2} \cdot \frac{U_0}{\nu} = \frac{\alpha_i^2}{k^2 - m^2\alpha^2} \cdot \frac{U_0}{\nu},$$

whence

$$k^2 = (m^2 + 1)\alpha^2.$$

Then, the point $\bar{x}_0$ attached to the pulsation $m\alpha U_0$ where β becomes imaginary will be such that

$$\alpha_i^2 \frac{U_0}{\nu \bar{x}_0} = (m^2 + 1)\alpha^2 = (m^2 + 1)\frac{\alpha_i^2 U_0}{\nu x_c} \quad \text{whence} \quad \bar{x}_0 = x_c \cdot \frac{1}{1 + m^2}.$$

Specifically, for $m = 1$, we obtain $\bar{x}_0 = \frac{x_c}{2}$ and, for $m = 0$, $\bar{x}_0 = x_c$.

The points $x_{1,2}$ where the velocity of propagation $\mathfrak{A}_1$ vanishes, will always be given by the above-indicated relations so that

$$\frac{x_1}{x_c} = \frac{\alpha^2}{k^2} - \left(\frac{\nu}{U_0}\right)^2 k^2 = \frac{1}{1+m^2} - \left(\frac{\nu}{U_0}\right)^2 \alpha^2 (1+m^2), \qquad \frac{x_2}{x_c} = \left(\frac{\nu}{U_0}\right)^2 \alpha^2 (1+m^2).$$

32. Study of the Second Approximation /117

The study of the first approximation, where $\alpha_1^2 \frac{U_0}{\nu x}$ had been assumed as constant in each integration step, can be supplemented by that of a second approximation in which, while making a step Δx from x_0, the variation $-\alpha_1^2 \frac{U_0}{\nu x_0^2} \Delta x$ is taken into consideration. Appendix III gives the corresponding developments, indicating that the propagation characteristics demonstrated for the first approximation are encountered also in the second. Finally, the complementary terms of the second approximation can be neglected for those of the first approximation.

33. Integration by Parts; Calculation of the Mean and Harmonic Component Terms

The above study, described in Section 31.1, led to a solution in which the function $f(x, t)$ of the perturbation stream function was expanded in narrow domains Δx_0 about each segment x_0, in the form of

$$f(x_0 + \Delta x_0) \cong 2\bar{f}_{1_0} \sin[\bar{\beta}(x_0)(\Delta x_0 - U_0 t_0) - \Phi] + f_{3_0},$$
$$f'_x(x_0 + \Delta x_0) \cong 2\bar{f}_{1_0} \bar{\beta}(x_0) \cos[\bar{\beta}(x_0)(\Delta x_0 - U_0 t_0) - \Phi].$$

where t_0 is some determined instant:

$$\bar{\beta}(x) = \sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x}} = k\sqrt{1 - \frac{\bar{x}}{x}}$$

[since $\bar{x}$ is defined by $\bar{\beta}(\bar{x}) = 0$].

Here, $\bar{f}_{1_0}$, f_{3_0} and thus also Φ are constants in the small narrow domain Δx_0 in question; however, they develop slowly from domain to domain about x_0, x_1, ..., etc. Consequently, these are unknown functions of x which must be determined at least approximately.

For this, we will stipulate that continuity of $f(x)$ and of its derivative exist when passing from one domain to the other, after which we will attempt to carry out an approximate integration by parts of the problem.

Thus, by setting $x_0 + \Delta x_0 = x_1$, the above-mentioned solution will be identified with the solution of the same form written in x_1 (where $\Delta x_1 = 0$) but for which

$$\bar{f}_1(x_1) = \bar{f}_1(x_0) + \bar{f}'_{1x}(x_0) \cdot \Delta x_0$$

which will be written as

$$\bar{f}_{1_1} = \bar{f}_{1_0} + \bar{f}'_{1_{0x}} \cdot \Delta x_0.$$

Similarly, /118

$$f_{2_1} = f_{2_0} + f'_{2_{0x}} \Delta x_0, \qquad \Phi_1 = \Phi_0 + \Phi'_{0_x} \Delta x_0, \qquad \bar{\beta}_1 = \bar{\beta}_0 + \bar{\beta}'_{0_x} \Delta x_0,$$

whence

$$f(x_0) + \Delta x_0 \cdot f'_x(x_0) \cong 2(\bar{f}_{1_0} + \bar{f}'_{1_{0x}} \Delta x_0) \sin[(\bar{\beta}_0 + \bar{\beta}'_{0_x} \Delta x_0)(-U_0 t_0) - (\Phi_0 + \Phi'_{0_x} \Delta x_0)]$$
$$+ f_{2_0} + f'_{2_{0x}} \cdot \Delta x_0 \quad \text{with} \quad f'_x(x_0) = 2\bar{f}_{1_0} \bar{\beta}_0 \cos[\bar{\beta}_0(-U_0 t_0) - \Phi_0],$$

$$f'_x(x_0) + \Delta x_0 \cdot f''_{x^2}(x_0) \cong 2(\bar{f}_{1_0} + \bar{f}'_{1_{0x}} \Delta x_0)(\bar{\beta}_0 + \bar{\beta}'_{0_x} \Delta x_0) \cos[(\beta_0 + \beta'_{0_x} \Delta x_0)$$
$$\times (-U_0 t_0) - (\Phi_0 + \Phi'_{0_x} \cdot \Delta x_0)] \quad \text{with} \quad f''_{x^2}(x_0) = -2\bar{f}_{1_0} \bar{\beta}_0^2 \sin[\bar{\beta}_0(-U_0 t_0) - \Phi_0].$$

By identifying (and linearizing) with respect to the variations in Δx_0, we readily obtain*

$$[\bar{\beta}'_{0_x} U_0 t_0 + \Phi'_{0_x} + \bar{\beta}_0] \cdot 2\bar{f}_{1_0} \cos[\bar{\beta}_0(-U_0 t_0) - \Phi_0] = 2\bar{f}'_{1_{0x}} \sin[\bar{\beta}_0(-U_0 t_0) - \Phi_0] + f'_{2_{0x}},$$

$$2\bar{f}_{1_0} \bar{\beta}_0 \sin[\bar{\beta}_0(-U_0 t_0) - \Phi_0] \cdot [\bar{\beta}_0 + \bar{\beta}'_{0_x}(U_0 t_0) + \Phi'_{0_x}]$$
$$= 2[\bar{f}'_{1_{0x}} \bar{\beta}_0 + \bar{f}_{1_0} \bar{\beta}'_{0x}] \cos[\bar{\beta}_0(-U_0 t_0) - \Phi_0].$$

The second equation is satisfied if we have

$$\bar{f}'_{1_{0x}} \cdot \bar{\beta}_0 + \bar{\beta}'_{0_x} \bar{f}_{1_0} = 0 \quad \text{and} \quad \bar{\beta}_0 + \bar{\beta}'_{0_x} U_0 t_0 + \Phi'_{0_x} = 0,$$

whence

$$\frac{\bar{f}'_{1_{0x}}}{\bar{f}_{1_0}} = -\frac{\bar{\beta}'_{0_x}}{\bar{\beta}_0},$$

i.e.,

$$\bar{f}_{1_0} = \frac{C}{\bar{\beta}_0} = \frac{C}{k\sqrt{1 - \frac{\bar{x}}{x}}} \quad \text{(C is a constant).}$$

Hence, we also obtain

$$\Phi'_{0_x} = -[\bar{\beta}'_{0_x} U_0 t_0 + \bar{\beta}_0],$$

i.e.,

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* It will be noted that $\sin[(\bar{\beta}_0 + \bar{\beta}'_{0_x} \Delta x_0)(-U_0 t_0) - (\Phi_0 + \Phi'_{0_x} \Delta x_0)] \cong$
$\cong \sin[\bar{\beta}_0(-U_0 t_0) - \Phi_0] + \Delta x_0 \cdot [\bar{\beta}'_{0_x}(-U_0 t_0) - \Phi'_{0_x}] \cos[\bar{\beta}_0(-U_0 t_0) - \Phi_0]$. The same holds for cos.

$$\Phi(x) - \Phi(x_0) = -U_0 t_0 k\left[\sqrt{1-\frac{\bar{x}}{x}} - \sqrt{1-\frac{\bar{x}}{x_0}}\right] - k\int_{x_0}^{x}\sqrt{1-\frac{\bar{x}}{x}}\,dx.$$

To exploit the first equation, it should be mentioned that it is always possible to make a choice of a time origin or - which comes to the same - of a time t_0 such that

$$\Phi_0 + \bar{\beta}_0 \cdot (U_0 t_0) = 0.$$

It is noted first that, in the step $x_1 - x_0$ where $f'_{3_{0x}} = 0$, the quantity f_2 /119 is a constant. In addition, at some time t, the term $f(x_0, t)$ will differ from $f(x_0, t_0) = f_3$ by the nonstationary term

$$-2\bar{f}_1(x_0)\sin[\bar{\beta}(x_0)U_0\tau]$$

when setting $\tau = t - t_0$. Here it is a question of a term oscillating harmonically about $f(x_0, t_0)$. We will denote $f(x_0, t_0)$ by $f_m(x_0)$ to indicate that a time average is involved here.

We can repeat the preceding reasoning, starting from the point x_1. However, to demonstrate the mean value $f_m(x_1)$, it is necessary to select a new time t_1 such that - as before -

$$\Phi(x_1) + \bar{\beta}(x_1) \,.\, U_0 t_1 = 0.$$

To pass from $f_m(x_0)$ to $f_m(x_1)$, it is thus necessary to calculate - starting with the time t_0 - a first variation of $f(x_0, t_0)$ with $\Delta x_0 = x_1 - x_0$, namely

$$|\Delta_1 f(x)|_{x_0}^{x_1} = 2\bar{f}_1(x_0) \cdot \bar{\beta}(x_0)\,\Delta x_0$$

and then a second variation, passing from the time t_0 to the time t_1:

$$|\Delta_2 f(x_1)|_{t_0}^{t_1} = 2\bar{f}_1(x_1) \cdot \bar{\beta}(x_1)\{-U_0(t_1 - t_0)\}.$$

We should note here that

$$\bar{\beta}(x_1)(-U_0 t_1) = +\Phi(x_1)$$

and

$$\bar{\beta}(x_1)(-U_0 t_0) = \bar{\beta}(x_1)\frac{\Phi(x_0)}{\bar{\beta}(x_0)}.$$

Thus, to obtain $f_m(x_1)$, it is necessary to increase $f_m(x_0)$ by

$$|\Delta f_m(x)|_{x_0}^{x_1} = 2\bar{f}_1(x_0) \cdot \bar{\beta}(x_0) \cdot \Delta x + 2\bar{f}_1(x_1) \cdot \bar{\beta}(x_1)\left[\frac{\Phi(x_1)}{\bar{\beta}(x_1)} - \frac{\Phi(x_0)}{\bar{\beta}(x_0)}\right].$$

On replacing $2\bar{f}_1(x) \cdot \bar{\beta}(x)$ by its value of

$$\frac{2C}{k\sqrt{1-\frac{\bar{x}}{x}}}k\sqrt{1-\frac{\bar{x}}{x}} = 2C,$$

we obtain, from x_0 to x_1,

$$|\Delta f_m(x)|_{x_0}^{x_1} = 2C\left[\Delta x_0 + \frac{\Phi(x_1)}{\bar{\beta}(x_1)} - \frac{\Phi(x_0)}{\bar{\beta}(x_0)}\right].$$

Similarly, from x_1 to x_2, /120

$$|\Delta f_m(x)|_{x_1}^{x_2} = 2C\left[\Delta x_1 + \frac{\Phi(x_2)}{\bar{\beta}(x_2)} - \frac{\Phi(x_1)}{\bar{\beta}(x_1)}\right]$$

and so on.

By term-by-term addition, we will obtain the variation of the time average of f(x) from x_0 to x, namely

$$|\Delta f_m(x)|_{x_0}^{x} = 2C\left[x - x_0 + \frac{\Phi(x)}{\bar{\beta}(x)} - \frac{\Phi(x_0)}{\bar{\beta}(x_0)}\right].$$

If, for the origin segment x_0, we select the segment $\bar{x}$ such that $\bar{\beta}(\bar{x}) = 0$, where we necessarily have $f(\bar{x}) \cong 0$ (no stationary constant and finite perturbation superimposed on the laminar field can exist), we finally obtain

$$\Phi(x) = -k\left[\int_{\bar{x}}^{x}\sqrt{1-\frac{\bar{x}}{x}}\,dx + U_0\bar{t}_0\sqrt{1-\frac{\bar{x}}{x}}\right],$$

where $\bar{t}_0$ represents the time such that

$$\frac{\Phi(\bar{x})}{\bar{\beta}(\bar{x})} = -U_0\bar{t}_0.$$

Since

$$\frac{\Phi(x)}{\bar{\beta}(x)} = -\frac{\int_{\bar{x}}^{x}\sqrt{1-\frac{\bar{x}}{x}}\,dx}{\sqrt{1-\frac{\bar{x}}{x}}} - U_0\bar{t}_0,$$

$f_m(x)$ will be given by

$$f_m(x) = |\Delta f_m(x)|_{\bar{x}}^{x} = 2C\left[x - \bar{x} - \frac{\int_{\bar{x}}^{x}\sqrt{1-\frac{\bar{x}}{x}}\,dx}{\sqrt{1-\frac{\bar{x}}{x}}}\right]$$

If the value of $f_m(x)$ is imposed in a segment x_c, then the constant C will result and one can simply calculate, in each segment x, $\bar{x} < x < x_c$, a function $\mu\left(\frac{x}{x_c}\right)$ such that

$$\mu\left(\frac{x}{x_c}\right) = \frac{f_m(x)}{f_m(x_c)} = \frac{x - \bar{x} - \dfrac{\int_{\bar{x}}^{x} \sqrt{1-\frac{\bar{x}}{x}}\, dx}{\sqrt{1-\frac{\bar{x}}{x}}}}{x_c - \bar{x} - \dfrac{\int_{\bar{x}}^{x_c} \sqrt{1-\frac{\bar{x}}{x}}\, dx}{\sqrt{1-\frac{\bar{x}}{x_c}}}}.$$

The accompanying table gives an example of the development of μ for $\bar{x} = 0.55\ x_c$. /121

$\frac{x}{x_c}$	0.55	0.60	0.70	0.90	1 —
$x - \frac{\bar{x}}{x_c}$	0	0.050	0.150	0.350	0.450
$\frac{x-\bar{x}}{x_c} - \frac{\int_{\bar{x}}^{x} \sqrt{1-\frac{\bar{x}}{x}}\, dx}{x_c\sqrt{1-\frac{\bar{x}}{x}}}$	0	0.025	0.055	0.105	0.125
$\mu\left(\frac{x}{x_c}\right)$	0	0.20	0.44	0.84	1

The expression for the nonstationary component will be

$$2\bar{f}_1(x) \sin \bar{\beta}(x)\, U_0\, \tau = \frac{2\, C}{k\sqrt{1-\frac{\bar{x}}{x}}} \sin k \sqrt{1-\frac{\bar{x}}{x}} \cdot U_0\, \tau,$$

using the notation τ defined at the beginning of this Section.

34. Perturbation in u′ Interior to the "Turbulent" Boundary Layer

We will limit the investigation of this case to the first approximation of f.

The fundamental condition (derived by Navier, see Sect.27)

$$\varphi''_{x^2}(f'_t + Uf'_x) + \varphi\,(f'''_{x^2t} + f'''_{x^3}U) + (g'''_{x^2t} + Ug'''_{x^3}) - U''_{y^2}(\varphi\, f'_x + g'_x)$$
$$= \nu\,[f\,\varphi''''_{y^4} + 2\,\varphi''_{y^2}f''_{y^2} + \varphi\, f''''_{x^4} + g''''_{x^4}]$$

where $U \cong \dfrac{U_0 - X}{\xi}\, y + X + u(y)$ will become, in this case, neglecting u(y) which is small with respect to the other terms and observing that X = const,

$$\varphi''_{y^2}\left[f'_t + \left(\frac{U_0 - X}{\xi}\, y + X\right)f'_x\right] + \varphi\left[f'''_{x^2t} + \left(\frac{U_0 - X}{\xi}\, y + X\right)f'''_{x^3}\right]$$
$$+ \left[g'''_{x^2t} + \left(\frac{U_0 - X}{\xi} + X\right)g'''_{x^3}\right]$$
$$\cong \nu\,[\varphi''''_{y^4}f + 2\,\varphi''_{y^2}f''_{x^2} + \varphi\, f''''_{x^4} + g''''_{x^4}].$$

Separating the independent terms of y from those containing this quantity /122 as factor, we obtain

$$\varphi''_{y^2}(f'_t + Xf'_x) + \varphi\,(f'''_{x^2t} + Xf'''_{x^3}) + g'''_{x^2t} + X\, g'''_{x^3}$$
$$= \nu\,[\varphi''''_{y^4}f + 2\,\varphi''_{y^2}f''_{x^2} + \varphi\, f''''_{x^4} + g''''_{x^4}],$$

which is the time equation, as well as

$$\varphi''_{y^2}f'_x + \varphi\, f'''_{x^3} + g'''_{x^3} = 0,$$

which is the space equation.

The first equation, taking the second equation into consideration, can also be written in the form

$$\frac{\partial}{\partial t}[\varphi''_{y^2}f + \varphi\, f''_{x^2} + g''_{x^2}] = \nu\,[\varphi''''_{y^4}f + 2\,\varphi''_{y^2}f''_{x^2} + \varphi\, f''''_{x^4} + g''''_{x^4}].$$

From the space equation, since φ is independent of x, we can derive

$$\frac{\varphi''_{y^2}}{\varphi} = -\frac{f'''_{x^3}}{f'_x} = k^2 \text{ const},$$

i.e.,

$$\varphi''_{y^2} - k^2\varphi = 0 \quad \text{and} \quad f'''_{x^3} + k^2 f'_x = 0.$$

Finally,

$$g'''_{x^3} = 0,$$

whence

$$\varphi = \varphi_1 e^{ky} + \varphi_2 e^{-ky}, \qquad f'_x = \mathring{i}k\,[f_1 e^{\mathring{i}kx} - f_2 e^{-k\mathring{i}x}].$$

Here, the integration is carried out directly rather than by parts.

Let us now return to the time equation. The derivatives of φ are of even order such that $\varphi_1 e^{ky} + \varphi_2 e^{-ky}$ can be factorized. For the term in $e^{\mathring{i}kx}$, it follows that

$$f'_{1_t}(k^2 - k^2) = \nu\,[k^4 - 2\,k^2 \cdot k^2 + k^4]\,f_1,$$

i.e., $0 = 0$ which means that $f'_{1_t}(t)$, $f_1(t)$ can be arbitrary quantities. The same holds for f'_{2_t}, $f_2(t)$ with respect to $e^{-\mathring{i}kx}$.

[Naturally, this is true only within the frame of the admitted approximation, with $u(y)$ being small. However, it follows from this that the conditions superimposed on $f_1(t)$, $f_2(t)$ are very weak and that these quantities will develop only slowly.]

Thus if, in an arbitrary segment x_c, the quantities f_1 and f_2 obey certain laws with respect to time, these laws will be conservative to values of $x > x_c$ relative to the "turbulent" layer.

It is also necessary to allow for

$$\frac{\partial}{\partial t} g''_{x^2} = \nu\, g''''_{x^4}, \qquad g'''_{x^3} = 0.$$

The second equation yields /123

$$g''_{x^2} = g_0(t),$$

where g_0 is constant with respect to x.

From the first equation we then obtain

$$g'_{0_t} = 0.$$

Thus, g_0 is a constant and $g'_x = g_0 x + g_1(t)$. The boundary conditions with respect to y will always be those concerning the wall

$$u'(0) = 0,$$

i.e.,

$$\varphi_2 = \varphi_1 = \frac{1}{2} \qquad \text{and} \qquad v'(0) = 0.$$

However,

$$v' = -[\mathring{i}k\,(f_1\, e^{\mathring{i}kx} - f_2\, e^{-\mathring{i}kx}) \cosh ky + g_0 x + g_1(t)],$$

$$u' = [(f_1\, e^{\mathring{i}kx} + f_2\, e^{-\mathring{i}kx} + f_3)\, k \sinh ky].$$

If several perturbations k are present, the condition $v'(0) = 0$ will lead to

$$\sum_k g_{0_k} = 0, \qquad \sum_k g_{1_k} = 0, \qquad \sum_k 2\,k f_{1_k} = 0$$

(since $f_2 = f_1$ so that v' can be real).

34.1 Reaction of the "Turbulent" Layer to an Imposed External Perturbation (in v')

It will be noted that the above expressions of v', u' are applicable to the perturbation inside the "stationary turbulent" layer which forms in response to an imposed external perturbation $v_0' \sin\alpha\,(x - U_0 t)$ of the type studied in the preceding Chapter (Sect.28).

However, the boundary conditions referring to this case must first be formulated.

Primarily, so that $u'v'$ be real, it is necessary that $f_1 = f_2$, i.e.,

$$v' = -[-2\,k f_1 \sin kx \cosh ky + g_0 x + g_1], \qquad u' = 2\,f_1 \cos kx \cdot k \sinh ky,$$

where $u'(0) = 0$ is thus satisfied. Here, $v'(0) = 0$ imposes

$$-2\,k f_1 \sin kx + g_0 x + g_1 = 0.$$

Consequently, we obtain

$$u'(\delta) = 2\,f_1 \cos kx \cdot k \sinh k\delta = 2\,f_{1_0} \cos k\,(x - U_0 x) \cdot k \sinh k\delta,$$

$$v'(\delta) = -[-2\,k f_1 \sin kx \cdot \cosh k\delta + g_0 x + g_1] = 2\,k\,f_1 \sin k\,(x - U_0 t)\,(\cosh k\delta - 1).$$

For the connectivity along $y = \delta$, at an external perturbation of the general type (see Sect.25, 26, 28) such as /124

$$u'_{\mathrm{ext}} = -\alpha\,F_0\,e^{-\alpha(y-\delta)} \cos\alpha\,(x - U_0 t),$$

$$v'_{\mathrm{ext}} = -F_0\,e^{-\alpha(y-\delta)}\,\alpha \sin\alpha\,(x - U_0 t) + v'_0 \sin\alpha\,(x - U_0 t),$$

it is necessary that

$$k = \alpha, \qquad 2\,f_{1_0} = -\frac{F_0}{\sinh k\delta}$$

whence

$$-F_0\,\alpha\,\frac{\cosh\alpha\delta - 1}{\sinh\alpha\delta} = -F_0\,\alpha + v'_0,$$

so that

$$\alpha\,F_0 = v'_0 \cdot \frac{\sinh\alpha\delta}{1 - (\cosh\alpha\delta - \sinh\alpha\delta)}, \qquad 2\,f_{1_0}\,\alpha = v'_0 \cdot \frac{1}{\cosh\alpha\delta - \sinh\alpha\delta - 1}.$$

As in the case of a laminar boundary layer, the reaction of the "turbulent" layer will include the appearance of a harmonic internal perturbation and that of a secondary external perturbation (tending rapidly to zero for y increasing beyond δ).

However, no limiting condition occurs here that might lead to impossibilities or contradictions with the approximations established on the basis of the calculation, i.e., to a limitation of the extent of the obtained solution. (No limiting hypothesis as to pressure. In addition, the terms in U''_{y^2} of the permanent flow are very weak and do not lead to exponentials that diverge from Y.)

CHAPTER VI /125

APPLICATION TO THE CONNECTION BETWEEN TURBULENT AND LAMINAR STATES; TRANSITION

35. General Remarks

We will attempt here to make use of the above-obtained results for defining the phenomena connected with the fact that the laminar state stops being possible starting from a critical segment x_c, as soon as a source of perturbations exists in the ambient medium.

Let us recall that x_c had been defined by the investigations in Chapter IV (Sect.29) concerning the action of an external harmonic perturbation applied in v', with a pulsation of αU_0 (with respect to time). It was demonstrated there that this perturbation caused the appearance, in the laminar boundary layer, of a corresponding perturbation in v' (and in rotations ω') which was harmonic and had a pulsation of αU_0 and that, in $x_c = \frac{\alpha_i^2}{\alpha^2} \cdot \frac{U_0}{\nu}$, the invariance of pressure characteristic for the laminar state no longer was able to exist.

This property is responsible for the fact that the laminar Blasius state can no longer be maintained and that it becomes necessary to pass to a different state constituting a second solution of the Navier-Stokes equations which had been derived in Part I under the designation of "stationary turbulent" state, indicating there that the definition of a time average was involved.

The "turbulent" state in question which must be acquired in an abscissa $(x_c + \delta x_c)$ extremely close to x_c (as demonstrated before), thus appears as a state perturbed with respect to the laminar state in the same segment, meaning that passage from the laminar to the "turbulent" state takes place over consideration of a field perturbation u' supplementary to the perturbation in v' which had given rise to the phenomenon in question. Thus, this alteration will be designated as second perturbation u'_{II}, a subject which will be studied in somewhat more detail.

36. Definition of the Family of Perturbations Permitting Connectivity between the Two States

To have the Blasius field $U(y)$ give way to the "stationary turbulent" field defined above, it is necessary that a family of perturbations u', of the type described in Chapter V, becomes superposed in the segment $x_c + \delta x_c$ such that their cumulative time averages will make up the difference existing between the two fields $(U_T - U_L)$ at each level y (Fig.23). /126

Each elementary perturbation is characterized by its constant such that (see Chapt.V, Sects.31.1 and 34)

$$u'_{II}(y) = \sum_m k_m \cdot \left(\varphi_{1_m} e^{k_m y} - \varphi_{2_m} e^{-k_m y}\right) \cdot f_m(x, t).$$

We have demonstrated before that only the harmonic perturbations can remain finite since the pulsations are $\overline{\beta} U_0$, where

$$\overline{\beta}_m = \sqrt{k_m^2 - \alpha_l^2 \frac{U_0}{\nu x}}.$$

Thus, $\overline{\beta}_m$ is an ascending function of x. This means that, in $x = x_c$, the quantity $\beta_m(x_c)$ must be inferior to or at most equal to α, since otherwise a value $x_c^* < x_c$ would exist where $\overline{\beta}_m(x_c^*)$ would assume the value α. This would mean that the point x_c^* rather than x_c would be the critical point at which the laminar state would have to be replaced by the "turbulent state", which is contrary to the hypothesis established. Thus, the maximum value which β_m can assume in x_c is α, so that

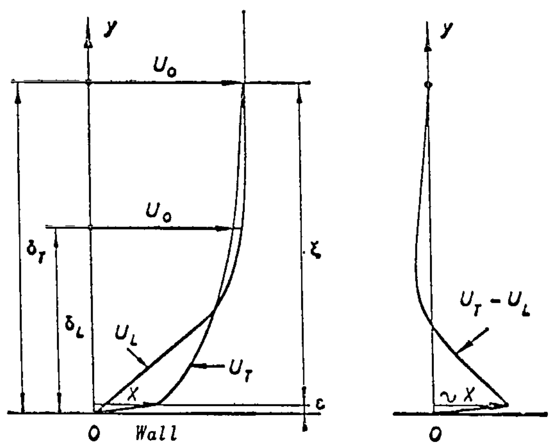

Fig.23

$$k_m^2 - \alpha_l^2 \frac{U_0}{\nu x_c} \leqslant \alpha^2,$$

i.e., since

$$\alpha_l^2 \frac{U_0}{\nu x_c} = \alpha^2, \qquad k_m^2 \leqslant 2\alpha^2.$$

The minimal value to be considered for $\overline{\beta}_m$ is $\overline{\beta}_m = 0$, corresponding to the case in which the perturbation ceases being harmonic in x_c.

Consequently, /127

$$k_m^2 - \alpha_l^2 \frac{U_0}{\nu x_c} \geqslant 0 \qquad \text{or} \qquad k_m^2 \geqslant \alpha^2.$$

Between these limits, all values are possible, i.e., values such as

$$\sqrt{k_m^2 - \alpha_l^2 \frac{U_0}{\nu x_c}} = m\alpha \qquad \text{or} \qquad k_m^2 = (1 + m^2)\alpha^2.$$

In an arbitrary x, we have $x < x_c$:

$$\overline{\beta}_m(x) = \sqrt{(m^2 + 1)\alpha^2 - \alpha^2 \frac{x_c}{x}} = \alpha \sqrt{m^2 + 1 - \frac{x_c}{x}}.$$

Here, $\bar{x} = x$ corresponds to $\bar{\beta}_m = 0$ such that

$$\frac{\bar{x}}{x_c} = \frac{1}{1+m^2}.$$

Again, $x = x'_c$ corresponds to $\bar{\beta}_m = \alpha$, such that

$$\frac{x_c}{x'_c} = m^2 \quad \text{or} \quad \frac{x'_c}{x_c} = \frac{1}{m^2}.$$

Consequently,

If $m > 1$, we have

$$\frac{x'_c}{x_c} = \frac{1}{m^2} < 1 \quad \text{and} \quad \frac{\bar{x}}{x_c} = \frac{1}{1+m^2} < 1.$$

On the other hand,

$$\frac{\frac{\bar{x}}{x_c}}{\frac{1}{2}} = \frac{2}{1+m^2} < 1,$$

i.e.,

$$\frac{\bar{x}}{x_c} < \frac{1}{2}.$$

If $m < 1$, we have

$$\frac{x'_c}{x_c} > 1 \quad \text{and} \quad \frac{\bar{x}}{x_c} < 1,$$

but

$$\frac{\frac{\bar{x}}{x_c}}{\frac{1}{2}} > 1,$$

so that /128

$$\frac{x_c}{2} < \bar{x} < x_c < x'_c.$$

So far as the development of pulsations with x is concerned, these cases are shown in the accompanying diagrams (Figs.24 and 25):

$$\bar{\beta} = m\,\alpha, \qquad m < 1,$$

$$\bar{\beta} = m\,\alpha, \qquad m > 1.$$

The latter case is impossible for the above-indicated reasons; consequently, no component u'_{II} can exist for which $m > 1$ since this would result in $x'_c < x_c$; the critical point x'_c where the state would necessarily stop being laminar under the action of the perturbation applied from the exterior at a pulsation αU_0 would be upstream of the point x_c, which is in contradiction with the stipulated conditions.

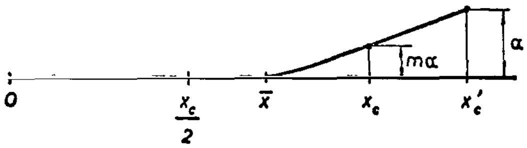

Fig.24

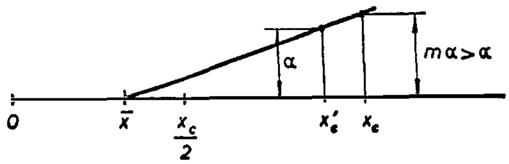

Fig.25

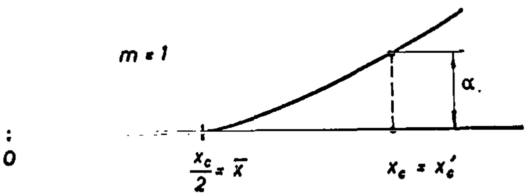

Fig.26

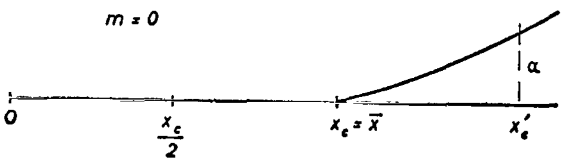

Fig.27

Conversely, $0 \leq m \leq 1$ represents possible evolutions whose limiting states are shown in the accompanying diagrams (Figs.26 and 27).

Thus, the component of the second perturbation u'_{II}, which furnishes a response of pulsation $m\alpha U_0$ ($m \leq 1$) in x_c, is generated at a point $\bar{x}$ downstream of $\frac{x_c}{2}$, the point where a response of pulsation αU_0 in x_c is generated. Consequently, the general expression of the second perturbation has the form /129

$$u'_{II} = \sum_{m=0}^{1} \alpha\sqrt{m^2+1}\left(\varphi_{1_m} e^{\alpha\nu\sqrt{m^2+1}y} - \varphi_{2_m} e^{-\alpha\nu\sqrt{m^2+1}}\right) f_m(x),$$

where the laminar existence domain is $\frac{x_c}{2} < x < x_c$.

37. Boundary Conditions

Here, we have to do with the particular form of the connectivity field $(U_T - U_L)$ in the segment x_c which tends to zero as $Y \to \bar{\xi} + \bar{\varepsilon}$ and which contains a discontinuity at the border $\bar{\varepsilon}$ of the sublayer.

At the wall, it is necessary that $u'_{II} = 0$ which leads to setting $\varphi_{1_m} = \varphi_{2_m}$ in the perturbation expression (see Sect.31)

$$u'_{II} = \sum_{m} k_m\left(\varphi_{1_m} e^{k_m\nu} - \varphi_{2_m} e^{-k_m\nu}\right) f_m(x, t).$$

Consequently, an expansion in sinh $k_m Y$ in the thickness of the sublayer ($0 < < Y < \bar{\varepsilon}$) must be used.

Since this thickness is very slight, it is possible to limit the expansion to terms linear in Y by setting $\sinh k_m Y \cong k_m Y$ such that, at the interface between sublayer and actual boundary layer, we obtain

$$u'_{\mathrm{II}_{\bar{\varepsilon}}} = \sum_m \varphi_{0m} k_m \sinh_m k_m \bar{\varepsilon} \cdot f_m(x, t) \cong \bar{\varepsilon} \cdot \sum_m \varphi_{0m} k_m^2 f_m(x, t).$$

Above this interface ($\bar{\varepsilon} < Y < \bar{\varepsilon} + \bar{\xi}$, i.e., $0 < y < \bar{\xi}$) a new perturbation in u'_{II} must be considered, which satisfies the above-investigated conditions (specifically, the Navier conditions) having all properties stipulated above and thus being of the form

$$u'_{\mathrm{II}} = \sum_m k_m \left(\varphi_{1_m} e^{k_m Y} - \varphi_{2_m} e^{-k_m Y}\right) f_m(x, t).$$

Consequently, the terms k_m and f_m are the same as those given above, except that the boundary layers with respect to y now are as follows: /130

For $Y \to \xi + \varepsilon$, the quantity u'_{II} will tend to zero (no matter what x and t might be). Consequently, it is necessary that $\varphi_1 = 0$ and we will set $\varphi_2 = 1$ (since this forms a product with f_m, nothing will be changed in the generality of the solution). Hence,

$$u'_{\mathrm{II}} = -\sum_m k_m e^{-k_m Y} f_m(x, t).$$

For $Y = \varepsilon$, the quantity u'_{II} should be linked with $u'_{\mathrm{II}_\varepsilon}$ of the sublayer. This fact, as will be demonstrated below, makes it possible to determine the thickness of the sublayer in each segment $x (x < x_c)$.

These observations imply:

1) For $x = x_c$, the quantity u'_{II} is such that, at the time average and at each level y, it will represent the deviation of the Blasius (laminar) field $U_L(y)$ from the "stationary turbulent" field $U_T(y)$, as defined in Part I.
2) For $x = \bar{x}_{Q_m}$, the corresponding component of u'_{II} in m will vanish.

37.1 Equations Defining the Formation of Perturbations u'_{II}

The complete expression of u'_{II}, allowing for the boundary conditions with respect to y and also considering the expression of $f_m(x)$, will be as follows in the actual boundary layer:

$$u'_{\mathrm{II}} = \sum_{m=0}^{1} \alpha \sqrt{m^2 + 1} \cdot e^{-\alpha\sqrt{1+m^2} \cdot y} \cdot f_m(x, t).$$

We must express now that the connectivity in x_c is such that, at the time average, we have

$$[u'^{(x)}_{11}]_{mean} = [U_T - U_L]_{x_0} \qquad \text{at each level } y,$$

i.e.,

$$\frac{U_T(y) - U_L(y)}{\alpha} = \sum_{m=0}^{1} \sqrt{m^2+1} \cdot e^{-\alpha\sqrt{1+m^2}\cdot y} \cdot [f_m(x_c)]_{mean}.$$

This represents an equation where the terms $[f_m(x_c)]_{mean}$ will be unknowns.

This can be written with a continuous variation of m such that $0 < m \leq 1$:

$$\frac{U_T(y) - U_L(y)}{\alpha} = \int_{m=0}^{1} \sqrt{m^2+1}\, e^{-\alpha\sqrt{m^2+1}\cdot y} f_m(x_c)\, dm.$$

Its solution is performed by a finite difference method, which makes it necessary to set /131

$$m = n \cdot \delta m,$$

where δm is a unitary variation and n is the rank of this variation.

We will take $0 < n < N$ where N corresponds to

$$N \cdot \delta m = 1$$

(so that $0 < m < 1$). Here, N is arbitrary but is taken sufficiently large so that $\delta m = \frac{1}{N}$ will be sufficiently low.

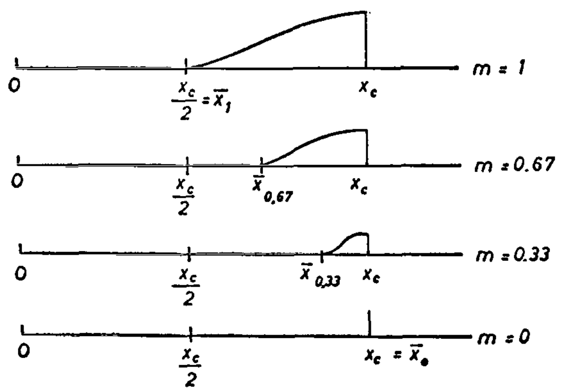

Fig.28

At each level y_i, the above equation yields

$$\left[\frac{U_T(y_i) - U_L(y_i)}{\alpha}\right]_{x_0} = f_{(0)}(x_c)\cdot \exp(\alpha\, y_i) + f_{(1)}\sqrt{1+\delta m^2}\cdot \exp\left(a\sqrt{1+\delta m^2}\cdot y_i\right)$$
$$+ f_{(2)}\sqrt{1+4\,\delta m^2}\cdot \exp\left(x\sqrt{1+4\,\delta m^2} y_i\right)$$
$$+ \dots + f_{(m)}\sqrt{1+\delta m^2\cdot n^2}\cdot \exp\left(\alpha\sqrt{1+n^2\,\delta m^2}\cdot y_i\right)$$
$$+ \dots + f_{(N)}\sqrt{2}\cdot \exp\left(\alpha\sqrt{2}\, y_i\right).$$

This is an equation of (N + 1) unknowns $f_{(0)}(x_c)$, $f_{(1)}(x_c) \dots f_{(m)}(x_c)$, ... $f_{(N)}(x_c)$. Repeating this equation for N + 1 levels y_i, we will obtain a Cramer system defining the unknowns.

The previously given relations defining $f_m(x) = \mu f_m(x_c)$ (Chapt.V, Sect.33) that specify their evolution with x can then be applied to each subscript (n), from which the perturbation u'_{II} will be known (at each value of x for $\frac{x_c}{2} < x < x_c$ and at each level y) as a mean component and as a harmonic component. It will be noted that, as a function of $m = n\delta m$, the domain of extent of each elementary perturbation m will differ from that of its neighbors and that the quantities $f_{(m)}(x)$ will have evolution diagrams of the type given in Fig.28.

At each point $\overline{x}_0 = \frac{x_c}{1 + m^2}$, the perturbation starts with a zero pulsation value and a zero time average. These increase with x, and the pulsation will reach a value of $\alpha U_0 m$ at $x = x_c$. /132

For $x > x_c$, where the turbulent state is definitely established, the structure (time average, nonstationary elongation, frequency) will remain practically invariant as demonstrated in Chapter V (Sect.34), under the condition that, in each segment x, the level y(x) corresponding to the level $y(x_c)$ of the critical segment is taken into consideration in accordance with the relation

$$\frac{y(x)}{y(x_c)} = \frac{\delta_T(x)}{\delta_T(x_c)}.$$

This results from the boundary conditions (with respect to y) to be satisfied, of the form given in Section 34.

38. Remarks on the Boundary with the Exterior, in the Transition Domain

In the domain of transition (where x is such that $\frac{x_c}{2} < x < x_c$), the border of the boundary layer is raised with respect to that of the original layer in such a manner that it connects, at the point x_c, with the border of the "turbulent" layer (Fig.29).

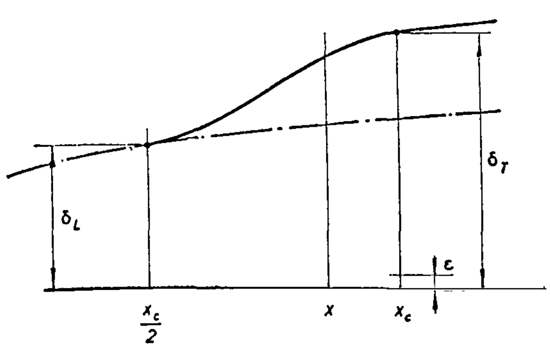

Fig.29

However, the notion of laminar boundary δ_L, as mentioned previously, is not specified by the laminar theory for which the Blasius function $U_L(y)$ is continuous at $0 < y < \infty$ (which is also true for the adopted image function).

The relations used above for determining U_L remain applicable to $y > \delta_L$, which means that also the Navier-Stokes equations and the resultant equations will remain valid.

Consequently, for studying the perturbation u'_{II}, it is unnecessary to distinguish between domains where $y < \delta_L$ and $y > \delta_L$. It is certain that if - with the aid of a given criterion - a value for $x = \frac{x_c}{2}$ can be assigned to $\delta_L\left(\frac{x_c}{2}\right)$, then an application of the same criterion will furnish, in the transition, a continuous border line connected to $\delta_T(x_c)$ in x_c (since all functions $f_{3_{om}}$ are continuous between $\frac{x_c}{2}$ and x_c, are zero in $\frac{x_c}{2}$, and are such that connectedness exists between the mean of u'_{II} and $U_T - U_L$ in x_c)*. /133

39. Domain of Transition; Experimental Comparison

According to the above analysis, the necessary passage to the stationary turbulent state in the critical segment $x_c = \frac{\alpha_i^2}{\alpha^2} \cdot \frac{U_0}{\nu}$ (under the influence of an external perturbation in v' of pulsation αU_0) takes place as a consequence of the continuous formation of a series of secondary harmonic perturbations in u'_{II} at the interior of the boundary layer along the segment $\frac{x_c}{2}$, x_c.

Their frequency is of the increasing type, attaining its maximum at x_c which corresponds to a pulsation αU_0.

These secondary perturbations contain a nonzero time average (connected, at each level y, to $U_T - U_L$ in x_c) as well as calculable nonstationary components which remain active beyond x_c, i.e., in the turbulent domain.

Because of this fact, it was possible to define a domain of transition extending, for the case of a plane plate, from $\frac{x_c}{2}$ to x_c as well as a turbulent state which justifies retaining its designation (permanence of a nonstationary state).

It will be found that the extremely simple result obtained in this manner [transition zone comprised between

$$\frac{\mathcal{R}_c}{2} = \frac{1}{2} \cdot \frac{\alpha_i^2}{\alpha^2}\left(\frac{U_0}{\nu}\right)^2 \quad \text{and} \quad \mathcal{R}_c = \frac{\alpha_i^2}{\alpha^2} \cdot \left(\frac{U_0}{\nu}\right)^2]$$

can be verified by practical experiments of measuring the friction coefficient - because of the fact that turbulent friction is much higher than laminar friction. Thus, the transition zone is characterized by the zone of abrupt increase of the coefficient of friction.

---

* We already mentioned (Chapt.IV, Sect.28) that, for the normal component V_L, the problem presents itself in a different manner because of the fact that the function V_L of the Blasius theory leads to an anomaly for $y \to \infty$. Thus, whenever it is a question of making a study of V, it is necessary to implicitly introduce a limit δ_L. For a study of U_L, as it is in question here, this restriction is unnecessary.

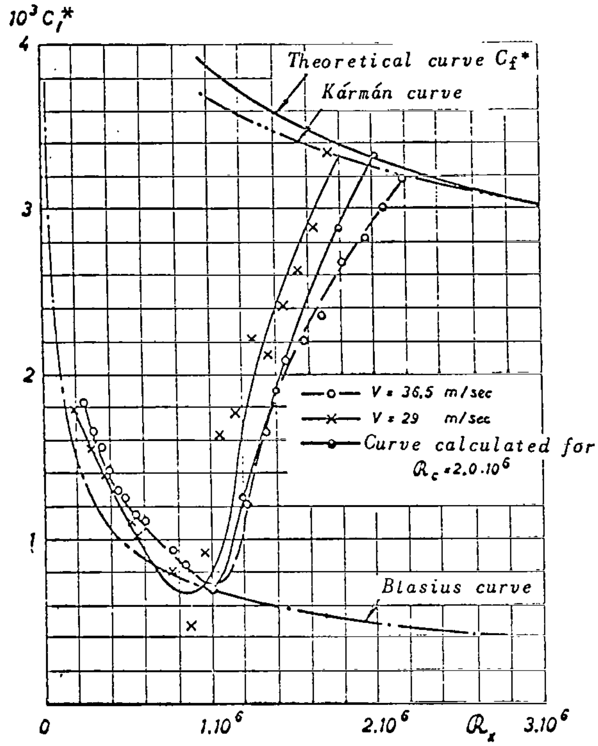

Fig.30 (Diagram III).

We will list here the early results of experiments made by Guienne (Ref.3) at the I.M.F.L.

Below, we are reproducing the pertaining diagrams $C_f^*(\mathfrak{R})$; Diagram III, Fig.30.

It seems that the agreement with the theoretical result mentioned above is of special interest, since the measurements were made on average states but are always somewhat interfered with and are rendered less accurate by the presence of subjacent nonstationary /134
components. The diagram also shows the slight deviation of the asymptotic (local) theoretical curve of $\overline{C}_f^*$ from the Kármán curve; finally, as previously stated, the quantity C_{fc}^* of the end of transition more or less coincides with the asymptotic quantity C_f^* (see Sect. Sect.21.3).

39.1 Diagram III

The local friction coefficient $C_f^*(\mathfrak{R}_x)$ is given in Fig.30.

The experiments by Guienne (Ref.3, p.13, Fig.7) are given here.

The comparison with the theory covers the following:

1) the asymptotic (local) curves C_f^* and the Kármán curves;
2) the term C_f^* of the end of transition coinciding with the asymptotic $\overline{C}_f^*$;
3) the transition extending from $\frac{1}{2}\mathfrak{R}_c$ to $\mathfrak{R}_c$.

40. Recapitulation of the Transition Calculations /135

Let us assume that the value ξ_c of the thickness of the actual boundary layer in x_c is known, which is the segment where the "turbulent" state is necessarily established.

In this case, the field $[U_T(y)]x_c$ is defined together with the velocity gradient at the wall $\left[\left(\frac{U}{y}\right)_0\right]_{x_c}^*$. This makes the following possible:

* For footnote see following page.

1) Establish the law $[U_T(y) - U_L(y)]_{x_c}$ and solve the system of equations of unknown quantities $f_{(m)}(x_c)$ where $0 < m \leq 1$ (see Sect.37), finally formulating the laws of development of each quantity $f_m(x)$ with x varying from $\frac{x_c}{2}$ to x_c, by calculating each function

$$\mu_m\left(\frac{x}{x_c}\right)=\frac{f_{(m)}(x)}{f_{(m)}(x_c)}$$

(see Sects.33 and 35, giving the calculation of the perturbation system generated from $\frac{x_c}{2}$ to x_c by the functional relation written in x_c

$$\left.[U_T(y)-U_L(y)]_{x_c}=\alpha\sum_{m=0}^{1}\sqrt{1+m^2}\,e^{-\alpha\sqrt{m^2+1}\,y}\,[f_{(m)}(x_c)]\right).$$

2) Calculate, in each segment x where $\frac{x_c}{2} < x < x_c$ and at each level y, the mean component of

$$[U_T(y)-U_L(y)]_x=\Delta U(y,x)=\sum_m\cdot\Delta U_m(y,x),$$

where

$$\frac{\Delta U_m(y,x)}{\Delta U_m(y,x_c)}=\mu_m\left(\frac{x}{x_c}\right).$$

3) Determine the velocity gradient at the wall, resulting from the same process:

$$\left[\left(\frac{\partial U}{\partial y}\right)_0\right]_x=\left[\left(\frac{\partial U_L}{\partial y}\right)_0\right]_x+\sum_m\left\{\left[\left(\frac{\partial U_{T_m}}{\partial y}\right)_0-\left(\frac{\partial U_L}{\partial y}\right)_0\right]_{x_0}\right\}\mu_m\left(\frac{x}{x_c}\right).$$

In this manner, the local friction coefficient will be fixed in each segment: /136

$$C_f^*=\frac{2\nu}{U_0^2}\left[\left(\frac{\partial U}{\partial y}\right)_0\right]_x.$$

The total friction coefficient is readily deduced from this by

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* Taking into consideration that the local friction coefficient at x_c is known:

$$\bar{C}_{f_c}^*=\frac{2\nu}{U_0^2}\left[\left(\frac{\partial U}{\partial y}\right)_0\right]_{x_c}, \quad \text{whence} \quad \left[\left(\frac{\partial U}{\partial y}\right)_0\right]_{x_c}.$$

$$C_f(x) = \frac{1}{x}\int_0^x C_f^*(x)\,dx = \frac{1}{x}\int_0^{\frac{x_c}{2}} C_{f_L}^*(x)\,dx + \frac{1}{x}\int_{\frac{x_c}{2}}^x C_f^*(x)\,dx$$

$$= \frac{x_c}{2x} C_{f_L}\left(\frac{x_c}{2}\right) + \frac{1}{x}\int_{\frac{x_c}{2}}^x C_f^*(x)\,dx.$$

Since, if the field is "turbulent" which is the case in $x = x_c$

$$\overline{C}_f(x) = 2\,\overline{\mathcal{A}} \cdot \frac{\overline{\xi}}{x},$$

the thickness of the turbulent layer will be such that

$$\overline{\xi}_c = \frac{x_c}{2\,\overline{\mathcal{A}}} C_f(x_c).$$

40.1 Calculation Procedure

The procedure can then be the following: To start the calculation, a provisional value of $\overline{\xi}_c$ is assigned with which all of the above-defined operations are performed after which $\xi_c \frac{x_c}{2\overline{\mathcal{A}}} C_f(x_c)$ is calculated. If there is a difference from the initially adopted value, the latter is corrected by half the difference and the calculation is resumed, and so on.

In fact, when making the numerical application, it will be found that only the very first terms $f_m(x_c)$ (those in which m is close to unity and for which $x_m \cong \frac{x_c}{2}$) will be notable and that only three of these terms for $\overline{x}_1 = \frac{x_c}{2}$, $\overline{x}_2 = 0.55\ x_c$, $\overline{x}_3 = 0.60\ x_c$ are necessary for completely taking care of the law $[U_T(y) - U_L(y)]_{x_0}$, irrespective* of the value of x_c.

Condensation of the domain of $\overline{x}_m$, in which the elements composing the perturbation about $\frac{x_c}{2}$ originate, then makes it possible - for calculating the development of $\Delta U(y, x)$ and $\left[\left(\frac{U}{y}\right)_0\right]_x$ - to use only one evolution law $\mu\left(\frac{x}{x_c}\right)$ corresponding to $\overline{x}_m \cong 0.55\ x_c$ $\left[\text{denoted by } \mu_0 \left(\frac{x_c}{x}\right)\right]$; see Section 33 and the table giving this law.

Since the local friction coefficient at $x = x_c$ where the flow is fully "turbulent" is $C_{f_c}^* = \overline{C}_{f_c}^*$, i.e., is sensibly the same as that of the asymptotic /137

* See Section 40.6, Diagram V, which shows the agreement of the imposed and image curves $\Delta U(y)$ with three coefficients corresponding to $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$.

solution (see Sect.21.3), it is easy to define the development of the local friction coefficient between $\frac{x_c}{2}$ and x_c. In fact, since the local friction is proportional to the velocity gradient U at the wall, we have

$$C_f^*(x) = C_{f_L}^*\left(\frac{x_c}{2}\right) + \mu_0\left(\frac{x}{x_c}\right)[\overline{C}_{f_0}^*(x_c) - C_{f_L}(x_c)].$$

Knowing the local friction at each point x, the total friction can be deduced from this as demonstrated above, permitting a determination of $C_f(x_c)$. Application of the relation of the turbulent state

$$\overline{C}_f(x_c) = 2\,\overline{\mathcal{A}}\cdot\frac{\overline{\xi}_c}{x_c}$$

will thus yield directly*

$$\overline{\xi}_c = \frac{x_c}{2\,\overline{\mathcal{A}}}\,\overline{C}_f(x_c).$$

The calculation can be continued until the development of the velocity field U(y) from $\frac{x_c}{2}$ to x_c is determined, as indicated at the beginning of this Section.

40.2 Decomposition of the Mathematical Operations for the Transition

The operations are as follows:

1) Fixation of the critical Reynolds number $\mathfrak{R}(x_c) = \mathfrak{R}_c$ as a function of the pulsation α and of the wavelength λ of the external perturbation:

$$\alpha = \frac{2\pi}{\lambda}, \qquad \mathcal{R}_c = \frac{\alpha_i^2}{4\pi^2}\cdot\frac{\lambda^2 U_0^2}{\nu^2} = \frac{\alpha_i^2}{\alpha^2}\cdot\frac{U_0^2}{\nu^2}$$

(where $\alpha_i \cong 0.375$ is the constant of the Blasius field).

2) Calculation, in the laminar regime, of the boundary layer thicknesses of the Blasius velocity fields at the points x_i $\left(\frac{x_c}{2} < x_i < x_c\right)$ retained for the calculation, followed by calculation of the coefficients of friction

$$y = \eta\sqrt{\frac{U_0}{\nu x}}, \qquad \delta_L = 5.5\sqrt{\frac{\nu x}{U_0}}, \qquad C_{f_L}^* = \frac{0.665}{\sqrt{\mathcal{R}}}, \qquad C_{f_L} = \frac{1.333}{\sqrt{\mathcal{R}}}, \qquad \text{etc.}$$

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* Here, $C_f(x_c)$ obviously is lower than the asymptotic friction (corresponding to a "turbulent" boundary layer starting from $x \cong 0$).

3) Determination of the characteristics of "turbulent" flow in the segment x_c. /138

a) Determination of the local (asymptotic) friction coefficient:

$$\bar{C}^*_{f_c} = \frac{8 \cdot 10^{-12} + \frac{700 \cdot 10^{-6}}{\mathcal{R}_c} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3.25}{\mathcal{R}_c^2}}{\left[8 \cdot 10^{-12} + \frac{700 \cdot 10^{-6}}{\mathcal{R}_c} + \frac{3.25}{\mathcal{R}_c^2}\right]^{\frac{3}{4}}}$$

and determination of the development of local friction between $\frac{x_c}{2}$ and x_c:

$$C_f^*(x) = \frac{0.665}{\sqrt{\mathcal{R}_x}} + \mu_0\left(\frac{x}{x_c}\right)[\bar{C}^*_{f_c} - C^*_{f_{L_c}}].$$

Calculation of the law of evolution of the total friction:

$$C_f(x) = C_{f_L}\left(\frac{x_c}{2}\right) \cdot \frac{x_c}{2x} + \frac{1}{x}\int_{\frac{x_c}{2}}^{x} C_f^*(x_c)\,dx$$

and, specifically, of $\bar{C}_f(x_c)$.

Hence,

$$\bar{\xi}_c = \frac{x_c}{2\bar{\mathcal{A}}}\bar{C}_f(x_c) \qquad \text{with} \qquad 2\bar{\mathcal{A}} = 0.226,$$

i.e.,

$$\mathcal{R}_{\bar{\xi}_c} = \frac{\bar{C}_f(x_c)}{2\bar{\mathcal{A}}} \cdot \mathcal{R}_c.$$

b) Determination of the velocity field at $x = x_c$ with $\frac{X}{U_0} = 0.45$:

$$\frac{U(y)}{U_0} \cong \frac{X}{U_0} + \left(1 - \frac{X}{U_0}\right)\frac{y}{\xi} + \sum_n \frac{\Phi_n}{U_0}\sin n\pi \cdot \frac{y}{\xi}$$

(where $\frac{1}{U_0}\Phi_n$ are the values calculated at the end of Sect.6.2).

The same holds for the determination of the velocity gradient at the wall, resulting from the value of $\bar{C}^*_f(x_c)$:

$$\left[\left(\frac{\partial U}{\partial y}\right)_0\right]_{x_c} = \frac{1}{2\frac{\nu}{U_0^2}} \cdot \bar{C}^*_{f_c}$$

with, for the laminar state,

$$\left[\left(\frac{\partial U_L}{\partial y}\right)_0\right]_{x_c} = \frac{1}{2\frac{\nu}{U_0^2}} \cdot \frac{0.665}{\sqrt{\mathcal{R}_c}}.$$

Formation of the quantities: /139

$$\Delta U(y, x_c) = [U_T(y) - U_L(y)]_{x_c} \quad \text{and} \quad \left[\Delta \left(\frac{\partial U}{\partial y}\right)_0\right]_{x_c}.$$

4) Application, in each segment x_i, $\frac{x_c}{2} < x_i < x_c$, of the following formulas:

$$[U(y)]_{x_i} = U_L(y)_{x_i} + \mu_0\left(\frac{x_i}{x_c}\right)[\Delta U(y, x_c)]$$

and

$$\left[\left(\frac{\partial U}{\partial y}\right)_0\right]_{x_i} = \left[\left(\frac{\partial U_L}{\partial y}\right)_0\right]_{x_i} + \mu_0\left(\frac{x_i}{x_c}\right)\left[\Delta\left(\frac{\partial U}{\partial y}\right)_0\right]_{x_c}$$

for the plot of the fields sought.

Below, we are giving tables and diagrams of the elements, to facilitate the numerical applications.

These diagrams refer to the calculation of the laws of evolution of the total (and local) friction, for values of $\mathfrak{R}_c$ 1×10^6, 2×10^6 and 3×10^6 relative to the transition.

The results are given in Diagrams IV of Section 40.5.

The result referring to the total friction incorporates the asymptotic law and the experimental points of Wieselsberger, Gebers, and Kempf (extracts from Prandtl, loc. cit.). The agreement is satisfactory.

As a typical example, we also determined the velocity fields for the case in which $\mathfrak{R}_c = 2.5 \times 10^6$.

The resultant diagrams will be compared with those given by Guienne (Ref.3); see the Diagrams VI in Section 40.7. The agreement is satisfactory for the velocity field and becomes excellent for the development of the boundary layer thicknesses in the transition.

40.3 Determinations of the Laws $C_f(\mathfrak{R})$, $C_f^*(\mathfrak{R})$ in the Transition (see Diagram IV)

First case: $\mathcal{R}_c = 1 \cdot 10^6$.

$\mathcal{R} \cdot 10^{-6}$	0.5	0.8	1	1.5	2	2.5	3	4	5	6	10	20
$\log \mathcal{R}$	5.7	5.9	6	6.17^5	6.30	6.40	6.47^5	6.60	6.7	6.78	7	7.3
Laminar $C_{f_L} \cdot 10^3$	1.88^5		1.33^1	1.1								
"Turbulent" asymptotic $\bar{C}_f\, 10^3$	5.63		5.06		4.40		4	3.75	3.56	3.42	3	2.62

(cont'd)

Transition $C_f^* \cdot 10^3$....	0.94	2.5	4.1	3.6	3,35	3.2	3.1	2.88	2.75	2.64	2,36	2,10
Transition (*by* integration) $C_f \cdot 10^3$.....	1.88^5	1.79	2.14	2.71	2.89	2.97^5	$2{,}98^5$	2.99^5	2.96	2,91	2.77	2.50
log $C_f \cdot 10^3$...........	0.275	0,252	0.33^2	0.43^2	0.46^1	0.47^3	0.47^4	$0{,}47^6$	$0{,}47^2$	0.46^4	0.44^2	0.39^8

Whence: $C_f(\mathcal{R}_c) = 2{,}14 \cdot 10^{-3}$, $\mathcal{R}_{\xi_c} = 9.5 \cdot 10^3$, $\dfrac{\overline{\xi}_c}{\delta_c} = 1{,}73.$

Second case: $\mathcal{R}_c = 2 \cdot 10^6$. /141

$\mathcal{R} \cdot 10^{-6}$.............	1	1.5	2	2.5	3	4	5	6	10	20
Transition $C_f^* \cdot 10^3$....	0.667	1.85	3.35	3.20	3.10	2.88	2.75	2.64	2.36	2.10
Transition $C_f \cdot 10^3$.....	1.33^5	1.31	1.62^7	1.96^5	2.16^5	2.36^5	2.46	2.50	2.49^5	2.35
log $C_f \cdot 10^3$...........	0.126	0.117	0.212	0.294	0.334	0.374	0.39	0.39^7	0.39^6	0.37

Whence: $C_f(\mathcal{R}_c) = 1.62^7 \cdot 10^{-3}$, $\mathcal{R}_{\xi_c} = 14.4 \cdot 10^3$, $\dfrac{\overline{\xi}_c}{\delta_c} = 1.84.$

Third case: $\mathcal{R}_c = 3 \cdot 10^6$.

$\mathcal{R} \cdot 10^{-6}$.............	1.5	2	2.5	3	4	5	6	10	20
Transition $C_f^* \cdot 10^3$....	0.550	1,05	2.20	3,10	2.83	2.75	2.64	2.36	2.1
Transition $C_f \cdot 10^3$.....	1.10	1.02	1.14^5	1.39^5	1,79	1.99	2.12^5	2.27^5	2,25
log $C_f \cdot 10^3$...........	0.04^2	0.01	0.06	0.14^4	0.25^3	0.30	0.32^7	0.36	0.35^5

Whence: $C_f(\mathcal{R}_c) = 1.39^5 \cdot 10^{-3}$, $\mathcal{R}_{\xi_c} = 18.5 \cdot 10^3$, $\dfrac{\overline{\xi}_c}{\delta_c} = 1.93^5.$

40.4 Calculation of the Transition for $\mathfrak{R}_c = 2.5 \times 10^6$

Development of the velocity field (see Diagrams VI):

$$C^*_{f_L}(x_c) = 0.422 \cdot 10^{-3}, \quad \overline{C}_f(x_c) = 3.2 \cdot 10^{-3} = 2\frac{\nu}{U_0^2}\left(\frac{\partial U}{\partial y}\right)_0 = 2\left[\frac{\partial \dfrac{U}{U_0}}{\partial\left(y\dfrac{U_0}{\nu}\right)}\right]_0$$

A value of $\left(\dfrac{\xi}{\delta}\right)_{x_c}$ = 1.88 is read from Diagram II.

Laminar Field

$\dfrac{y}{\delta}$..............	0	0.32^5	0.36^5	0.42^9	0.48^1	0.53^3	0.63^7	0.74	0.79^5	0.84^5	1
$\dfrac{U_L}{U_0}$............	0	0.58	0.65	0.74	0.80^1	0.85	0.92^7	0.97	0.98^5	0.99^1	1
$\mathcal{R}_c = 2.5 \cdot 10^6$ $\delta_L = 8.7\sqrt{\mathcal{R}} \cdot \dfrac{U_0}{\nu}$.											
$y\dfrac{U_0}{\nu} \cdot 10^{-3}$.....	0	2.82	3.17	3.73	4.19	4.63	5.50	6.45	6.91	7.35	8.70

(cont'd)

$\mathcal{R}_c = 2.25 \cdot 10^6$. $y \frac{U_0}{\nu} 10^{-3}$	0	2.77	3-	3.53	3.97	4.39	5.21	6.11	6.55	7-	8.25
$\mathcal{R}_c = 1.75 \cdot 10^6$. $y \frac{U_0}{\nu} 10^{-3}$	0	2.36	2.65	3.12	3.50	3.87	4.60	5.11	5.78	6.15	7.27
$\mathcal{R}_c = 1.25 \cdot 10^6$ $y \frac{U_0}{\nu} 10^{-3}$	0	2-	2.24	2.64	2.96	3.28	3.89	4.56	4.90	5.20	6.15

"Turbulent" field $\frac{\xi}{\delta} = 1.88$, $\mathcal{R}_c = 2.5 \cdot 10^6$. /144

$\frac{y}{\xi}$	0	0.16^7	0.25	0.33^3	0.50	0.66^7	0.75	0.83^5	1
$\frac{U_T}{U_0}$	$\sim 0,45$	$0,65^3$	$0,73^6$	$0,80^4$	$0,90^1$	0,95	0.97^3	0.99^3	1
$y \frac{U_0}{\nu} \cdot 10^{-3}$	0	2,74	4,10	5,28	8,20	10,9	12.3	13.7	16.3^5

Hence, for:

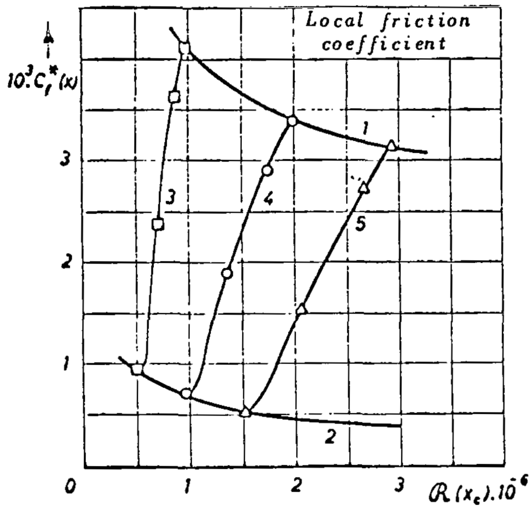
$y \frac{U_0}{\nu} \cdot 10^{-3}$	0	1	2	3	4	6	8	10	12	14	16
the values of: $\frac{\Delta U}{U_0}$	?	0.31	0.19	0.06	-0.04^5	-0.12	-0.10^5	-0.06	-0.03	-0.007	0
$\mathcal{R}_c = 2.25 \cdot 10^6$ $\mu = 0.84$. $\frac{\Delta U}{U_0}$		0.26	0.158	0.05	-0.03^7	-0.10	-0.09	-0.05	-0.02^5	-0.00^6	0
$\mathcal{R}_c = 1.75 \cdot 10^6$ $\mu = 0.45^5$. $\frac{\Delta U}{U_0}$		0.14	0.08^6	0.02^7	-0.02	-0.05^5	-0.04^7	-0.02^7	-0.01^3	-0.00^3	0

$\mathcal{R}_c = 2.5 \cdot 10^6$, $\left[\frac{\partial U_T}{\partial y} - \frac{\partial U_L}{\partial y}\right]_0 \cdot \frac{\nu}{U_0^2} \cdot 10^{-3} = 1.6 - 0.21 = 1.4$ with $\left(\frac{\partial U_T}{\partial y}\right)_0 \frac{\nu}{U_0^2} \cdot 10^{-3} = 1.6$

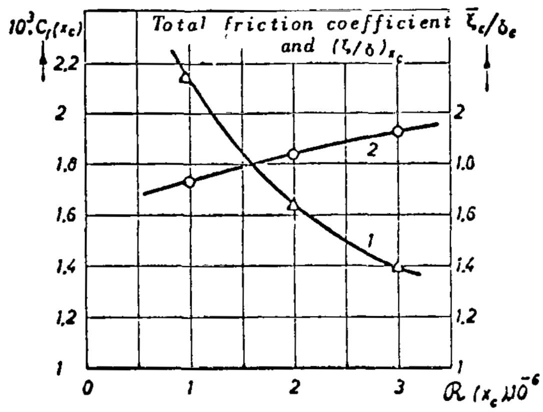
For $\mathcal{R} = 0.9\, \mathcal{R}_c = 2.25 \cdot 10^6$ $\left(\frac{\partial U}{\partial y}\right)_0 \frac{\nu}{U_0^2} \cdot 10^{-3} = 0.22^3 + 1.4 \cdot 0.84^5 = 1.39^5$

For $\mathcal{R} = 0.7\, \mathcal{R}_c = 1.75 \cdot 10^6$ $\left(\frac{\partial U}{\partial y}\right)_0 \frac{\nu}{U_0^2} \cdot 10^{-3} = 0.25^1 + 1.4 \cdot 0.45^5 = 0.90$

For $\mathcal{R} = 0.5\, \mathcal{R}_c = 1.25 \cdot 10^6$ $\left(\frac{\partial U}{\partial y}\right)_0 \frac{\nu}{U_0^2} \cdot 10^{-3} \cong 0.3.$

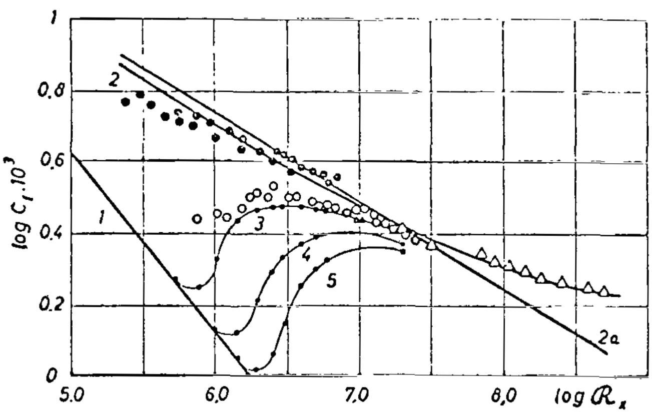


1 - Asymptotic "turbulent" friction
2 - Laminar friction
3, 4, 5 - Transitions for $\mathcal{R}_c = 1 \times 10^4$, 2×10^6, 3×10^6



1 - Friction coefficient at $x = x_c$
2 - Ratio of "turbulent" to laminar boundary layer thicknesses at $x = x_0$

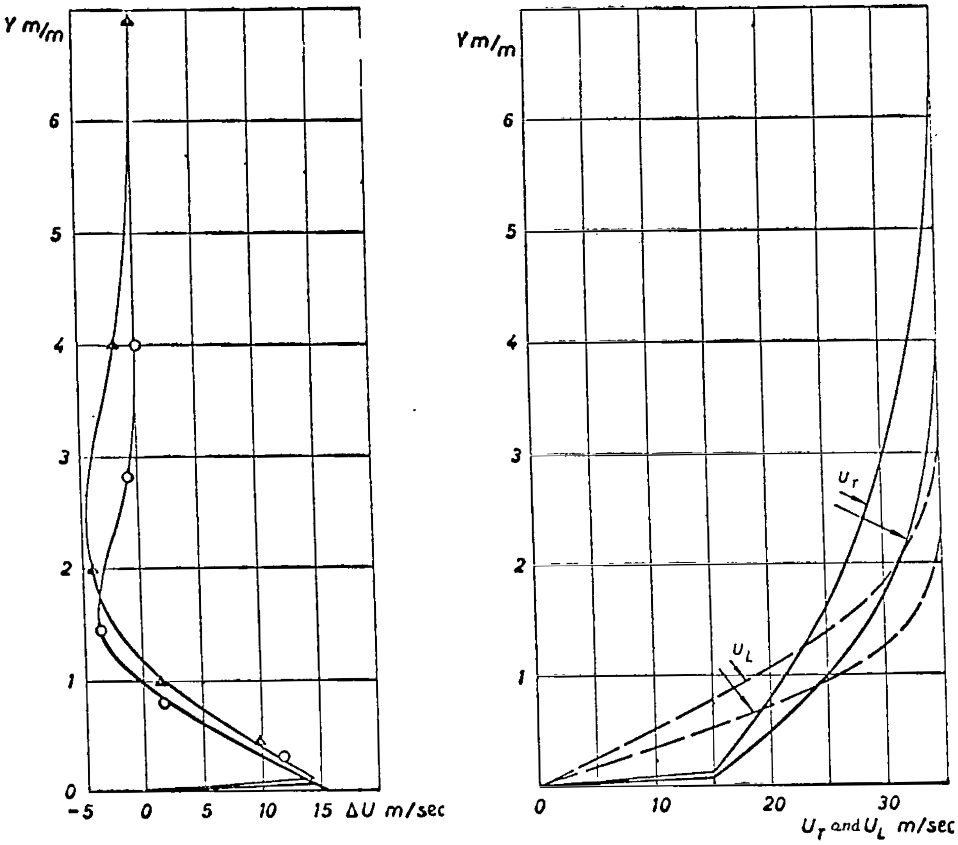
The above diagrams result from the figures given in the preceding tables.



1 - Laminar friction
2 - Asymptotic "turbulent" friction, $\bar{\beta}$ and $\bar{a}$ of experimental origin
2a - Asymptotic "turbulent" friction, $\bar{\beta}$ of theoretical origin $\bar{a} \equiv 0$
3, 4, 5 - Transitions; for $\mathcal{R}_c = 1 \times 10^6$, 2×10^6, 3×10^6 (integration of C^*_f)
●●● - Experimental points by Wieselsberger
○○○ - Experimental points by Gebers
△△△ - Experimental points by Kempf.

Note: The experiments by Gebers contained a laminar zone and thus a transition (Ref.2, p.151).

Fig.31 (Diagrams IV); Coefficient of Total Friction (Comparison with Experiment).



Determination of the coefficients $f_m(x_c)$ in the functional equation (see Sect. 39):

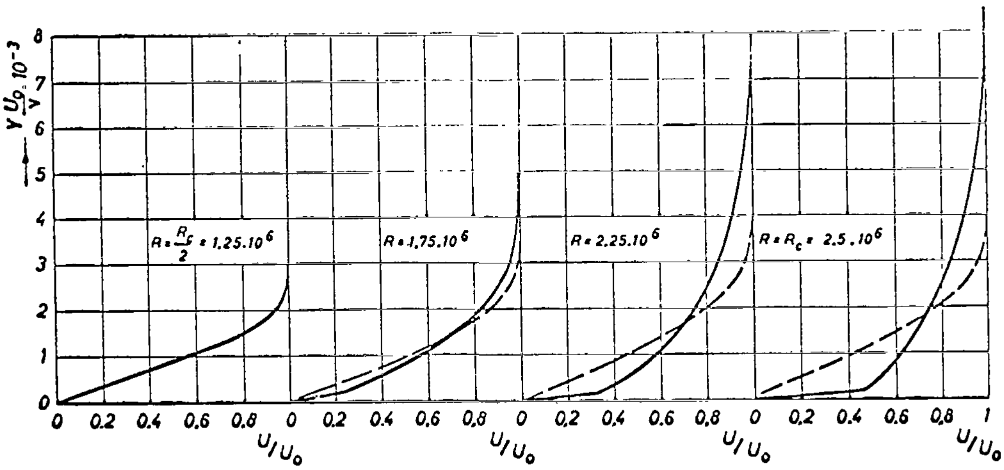
For $U_0 = 35$ m/sec; $[\Delta U(y)]_{x_0} = [U_T(y) - U_L(y)]_{x_0} = \sum_0^1 \alpha \sqrt{1+m^2}\, e^{-x\sqrt{1+m^2}\cdot y} f_m(x_c)$

$\frac{\xi_c}{\delta_c} = 1.78$: $\alpha f_1 = -12.1$, $\alpha f_2 = 10.4$, $\alpha f_3 = -1.60^*$ furnishing the points ○ of the diagram $[\Delta U(y)]$

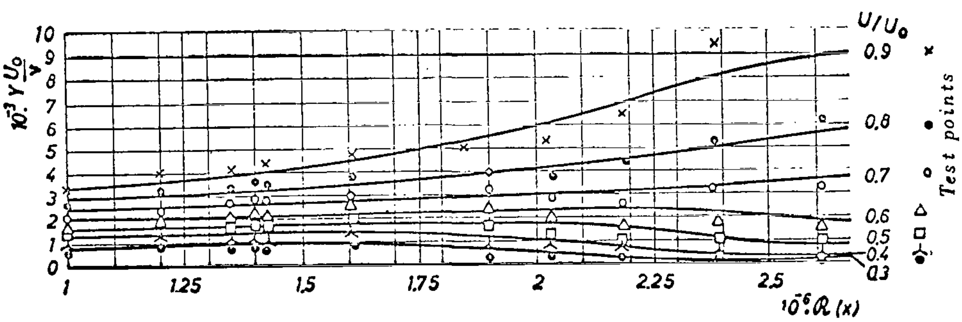
$\frac{\xi_c}{\delta_c} = 2.12$: $\alpha f_1 = -4.1$, $\alpha f_2 = 2.15$, $\alpha f_3 = 7.3$ furnishing the points Δ of the diagram $[\Delta U(y)]$

$\left(f_1 \sim \frac{\bar{x}}{x_c} = 0.5,\ m = 1; \quad f_2 \sim \frac{\bar{x}}{x_c} = 0.55,\ m = 0{,}00^* \quad f_3 \sim \frac{\bar{x}}{x_c} = 0.60,\ m = 0.81^* \right)$

Fig.32 (Diagrams V).



Development of the velocity field in the transition, Case $\mathcal{R}_c = 2.5 \cdot 10$
Segments of calculation in $\mathcal{R} \cdot 10^{-4} = 1.25 \cdot 1.75 \cdot 2.25 \cdot 2.50$



Calculated iso-velocity curves; $\mathcal{R}_c = 2.5 \cdot 10^6$

Fig.33 (Diagrams VI); Comparison with Experiments by Guienne.

41. Artificial Rearward Shift of the Transition

It has been shown previously (Sect.29) that, in the presence of an external perturbation of pulsation αU_0, the laminar state is unable to persist beyond the critical segment x_c, such that

$$x_c = \frac{\alpha_i^2}{\alpha^2} \cdot \frac{U_0}{\nu}.$$

In fact, starting from this point, the pressure gradients no longer are small (since they cease tending to zero with increasing y and actually diverge /145
rapidly). The hypotheses of invariance of pressure, which form the basis of the Blasius theory and which, from this theory, have taken the laminar field into consideration, stop being applicable. The laminar state necessarily is replaced by the other solution of the Navier-Stokes equations known as "stationary turbulent" solution which does not involve any hypothesis of invariance of pressure.

However, the divergence of the pressure gradients with y is due to the presence of the exponential $e^{\alpha-\alpha_i\sqrt{\frac{U_0}{\nu x}}y}$ which, in turn, is due to the exponential $e^{\alpha y}$ existing in the function $\varphi(y)$ which enters the stream function $\psi(x, y, t)$ of perturbation $\psi = \varphi(y)f(x, t) + g(x, t)$.

Finally, the boundary conditions at the wall $u'(0) = 0$, $v'(0) = 0$, imposed on the internal perturbation of the laminar layer formed in response to the external perturbation, are responsible for the exponential in question.

Let us recall the most general form of the velocity components of the internal perturbation (see Sect.28):

$$u'(y) = \varphi'_y \cdot f = \alpha(\varphi_1 e^{\alpha y} - \varphi_2 e^{-\alpha y}) f,$$

$$v'(y) = -\varphi f'_x - g'_x = -(\varphi_1 e^{\alpha y} + \varphi_2 e^{-\alpha y}) f'_x - g'_x.$$

Let us assume that the wall, which is no longer fixed, is induced to move with a harmonic deformation motion such that

$$y = y_0 \cos\alpha(x - U_0 t).$$

The normal velocity component $\frac{dy}{dt} = +y_0 U_0 \alpha \sin\alpha(x - U_0 t)$ must coincide with the new perturbation boundary condition, meaning that the following must be posed, at $y = 0$ which is the mean figure of the undulating wall with a pulsation αU_0:

$$v'(0) = y_0 U_0 \alpha \sin\alpha(x - U_0 t).$$

It is no longer necessary to stipulate $u'(0) = 0$ since $u'(0)$ also becomes a harmonic function of $\alpha(x - U_0 t)$, meaning that, instead of writing $\varphi_1 = \varphi_2$ as done above, we can now write $\varphi_1 = 0$, $\varphi_2 \neq 0$. We will set $\varphi_2 = 1$ (since φ always forms a product with f, nothing is changed in the generality of the discussion). Hence,

$$v'(y) = -e^{-\alpha y} f' - g'_x, \qquad u'(y) = \alpha e^{-\alpha y} f,$$

i.e., the conditions determining f and g are identical with those investigated previously (see Sects.28 and 29):

$$v'(y) = -(f_0 + g_0) \frac{e^{-\alpha y} - 1}{e^{-\alpha\delta} - 1} \alpha \sin \alpha (x - U_0 t),$$

$$u'(y) = -\alpha e^{-\alpha y}\left(\frac{f_0 + g_0}{e^{-\alpha\delta} - 1} \cdot \cos(\alpha U_0 t) + \text{const}\right).$$

Thus, in the expressions given in Section 29, the exponential $e^{\left(\alpha - \alpha_1\sqrt{\frac{U_0}{\nu x}}\right)y}$ vanishes; simultaneously, the causes of divergence also vanish so that the laminar state will disappear. /146

41.1 Wall Structure Satisfying the Preceding Conditions

Can one more precisely define how to cause the occurrence of a characteristic harmonic motion of the wall?

Let us assume that such a wall is formed by a membrane stretched over an elastic medium at constant pressure p_0 which, in turn, is resting on the fixed wall itself. Let us neglect the mass of the membrane. Then, at each instant, equilibrium exists between the forces applied to the membrane from the outside and its elastic reaction (Fig.34).

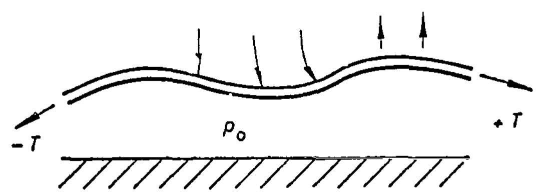


Fig.34

In a zone with a curvature radius R, let us consider a segment $Rd\theta$. The normal elastic component will be $Td\theta$. The normal component due to the exterior and interior pressures will be $\Delta p \cdot Rd\theta$, with

$$\frac{1}{R} = y''_{x^2} = -y_0 \alpha^2 \cos \alpha (x - U_0 t).$$

The condition $Td\theta = \Delta p R d\theta$ leads to

$$\Delta p = -y_0 T \cdot \alpha^2 \cos \alpha (x - U_0 t).$$

If the modulus of elasticity of the membrane is low, then $T \cong T_0$ where T_0 is a tension that is constant with respect to the deformation.

On the other hand,

$$\Delta p = p - p_0 = -\int_{x_0}^{x} \rho v' \cdot \frac{\partial U(0)}{\partial y} dx,$$

with

$$\frac{\partial U(0)}{\partial y} = U_0 \sum_i a_i \alpha_i \sqrt{\frac{\nu x}{U_0}}, \qquad v' = v'_0 \sin\alpha(x - U_0 t) = y'_0 U_0 \alpha \sin\alpha(x - U_0 t).$$

Then, /147

$$\Delta p \cong \rho \frac{v'_0}{\alpha} \cos\alpha(x - U_0 t)\, U_0 \sum_i a_i \alpha_i \sqrt{\frac{\nu x}{U_0}},$$

when considering that the term in $\sqrt{\frac{\nu x}{U_0}}$ develops very slowly* relative to the term in αx. Thus, the relations to be satisfied are as follows:

$$y_0 = \frac{v'_0}{U_0 \alpha}$$

and

$$\frac{\rho v'_0}{\alpha} \cos\alpha(x - U_0 t) \cdot U_0 \sum_i a_i \alpha_i \sqrt{\frac{\nu x}{U_0}} = -y_0 T_0 \alpha^2 \cos\alpha(x - U_0 t)$$
$$= -\frac{v'_0}{U_0} T_0 \alpha \cos\alpha(x - U_0 t).$$

Hence,

$$T_0 = \rho U_0^2 \cdot \frac{\sum_i a_i \alpha_i}{\alpha} \sqrt{\frac{\nu x}{U_0}}$$

---

* In fact, operating from a value x_0 taken as the origin of x, selected such that $\sin\alpha(x - U_0 t) = \sin\alpha(\Delta x - U_0 t)$ and integrating by parts twice in succession, we obtain

$$\int \sqrt{\frac{\nu x}{U_0}} \sin\alpha(x - U_0 t)\, dx = \int \sqrt{\frac{\nu(x_0 + \Delta x)}{U_0}} \sin\alpha(\Delta x - U_0 t)\, d\Delta x$$
$$= -\sqrt{\frac{\nu x_0}{U_0}} \cdot \frac{\cos\alpha(\Delta x - U_0 t)}{\alpha} + \frac{1}{2\alpha}\sqrt{\frac{\nu}{U_0 x_0}} \int \Delta x \cos\alpha(\Delta x - U_0 t)\, d\Delta x$$
$$= -\sqrt{\frac{\nu x_0}{U_0}} \cdot \frac{\cos\alpha(\Delta x - U_0 t)}{\alpha} - \frac{1}{2\alpha}\sqrt{\frac{\nu}{U_0 x_0}} \cdot \frac{\sin\alpha(\Delta x - U_0 t)}{\alpha}\Delta x + \frac{1}{2\alpha}\sqrt{\frac{\nu}{U_0 x_0}} \frac{\cos\alpha(\Delta x - U_0 t)}{\alpha}.$$

The last term can be neglected for the two first terms since $\frac{1}{\alpha}\sqrt{\frac{\nu}{U_0 x_0}}$ is very small with respect to $\sqrt{\frac{\nu x_0}{U_0}}$. In the two first terms, we identify

$$-\sqrt{\frac{\nu x}{U_0}} \cdot \frac{\cos\alpha(\Delta x - U_0 t)}{\alpha} = -\sqrt{\frac{\nu x}{U_0}} \cdot \frac{\cos\alpha(x - U_0 t)}{\alpha}.$$

which fixes the tension to be given to the membrane so that an external perturbation of pulsation αU_0 will not cause transition to the turbulent state. It is obvious that, in the solution obtained in this manner, the latter must be linked to the external perturbation and to the abscissa x.

These observations can be compared with recently reported findings on the properties of the skin of certain cetaceans (dolphins, porpoises, etc.). It is not entirely out of the question that skin muscles exist which adjust the tension of the skin to the frequency of perturbations of the ambient medium (at velocities of 20 - 30 m/sec, these frequencies attain kilocycles).

It is also conceivable to provide the walls of wings and fuselages with a skin of such type. /148

The analysis performed here does not reveal that the dissipative qualities of the substance forming the skin (permeable or not) play any role at all. Apparently, only the elastic characteristics are of significance*.

Experiments would have to be made to verify or nullify these provisional conclusions.

* We attempted to find some indications along this line by using a dissipative permeable skin, but were unsuccessful.

PART III

INTRODUCTION TO STUDIES OF THE COMPRESSIBLE STATE

CHAPTER VII /151

GENERAL REMARKS ON THE TURBULENT COMPRESSIBLE BOUNDARY LAYER AND HEAT TRANSFER

42. General Scheme

Let us return to the original schemes and to the hypotheses on the order of magnitude, established in Part I (Sect.1). However, here the specific mass ρ, the viscosity coefficient μ, the heat transfer coefficient k, and the Kelvin temperature T will be the variables used. Four new relations corresponding to these four new variables must then be introduced into the problem. Classically, these will be:

1) The equation of state of a perfect gas:

$$\frac{p}{\rho} = \mathrm{RT}$$

where $\mathrm{R} = \frac{p_0}{\rho_0 T_0} = \frac{\gamma - 1}{\gamma} \cdot gC_p$ (with the subscript 0 referring to the reference state); γ is the polytropic constant of the gas and gC_p its specific heat at constant pressure related to unit mass, with the heat being expressed in mechanical work units.

2) An experimental relation concerning the development of viscosity with temperature:

$$\mu = \mu_0 \left(\frac{\mathrm{T}}{\mathrm{T}_0}\right)^{\omega} \qquad (\omega \cong 0.8 \text{ for air}).$$

3) The energy relation (Ref.4) whose complete expression reads:

$$\rho g C_p \left[\mathrm{U}\frac{\partial \mathrm{T}}{\partial x} + \mathrm{V}\frac{\partial \mathrm{T}}{\partial y} \right] = \mathrm{U}\frac{\partial p}{\partial x} + \mathrm{V}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(k\frac{\partial \mathrm{T}}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial \mathrm{T}}{\partial y}\right)$$
$$+ \mu\left[2\left(\frac{\partial \mathrm{U}}{\partial x}\right)^2 + 2\left(\frac{\partial \mathrm{V}}{\partial y}\right)^2 + \left(\frac{\partial \mathrm{U}}{\partial y} + \frac{\partial \mathrm{V}}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{V}}{\partial y}\right)^2\right].$$

Let us recall that k is such that /152

$$\frac{d\mathrm{Q}}{dt \cdot d\mathrm{S}} = k\frac{\partial \mathrm{T}}{\partial n}.$$

Here, Q must be expressed in mechanical units.

4) Finally, since P is the Prandtl number such that

$$\mathrm{P} = \frac{\mu g C_p}{k} \qquad (0.7 \text{ for air})$$

the fourth relation reads

$$k = \frac{\mu g C_p}{P} = k_0 \left(\frac{T}{T_0}\right)^{\omega}$$

Here, γ, gC_p, P, ω are the absolute gas constants.

The previously used relations (equation of continuity and Navier-Stokes equation) will be written here as follows:

equation of continuity:

$$\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) = 0.$$

Navier-Stokes equation [in two-dimensional regime; see Schlichting (Ref.4, p.51)]:

$$\frac{\partial p}{\partial x} + \rho\left[U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial U}{\partial t}\right] = \frac{\partial}{\partial x}\left[\mu\left\{2\cdot\frac{\partial U}{\partial x} - \frac{2}{3}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right\}\right]$$
$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right];$$
$$\frac{\partial p}{\partial y} + \rho\left[U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \frac{\partial V}{\partial t}\right] = \frac{\partial}{\partial y}\left[\mu\left\{2\frac{\partial V}{\partial y} - \frac{2}{3}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right\}\right]$$
$$+ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\right].$$

As in the incompressible case, the velocity scheme will be formed by (see Fig.1):

1) a constant gradient, from the wall to the boundary U_1 of the sublayer ϵ;
2) a principal tangential component linear in y:

$$U = U_1 + \frac{\overline{U}}{\xi} y \qquad (\text{with } \overline{U} = U_0 - U_1)$$

in the actual boundary layer ξ. This component is attached to a small normal component V such that /153

$$(\rho U)'_x + (\rho V)'_y = 0;$$

3) a complementary tangential component u which is small compared to U [to which a very small v corresponds, with $(\rho u)'_x + (\rho v)'_y = 0$]. Introduction of this modification will permit satisfying the Navier-Stokes equations.

We will also consider ξ as principal, all derivatives with respect to x as small and of the second order, and $\frac{\mu}{\rho U_0 x}$ as very small so that the terms in

$\frac{\mu}{\rho x} \cdot \frac{u}{U_0}$ can be neglected while those in $\frac{\mu}{\rho x} \cdot \frac{U}{U_0}$ are retained.

43. Derivation of Equations

After this, we will return to the course followed in Part I (see Sects. 2 and 3).

43.1 Expressing V by the Condition of Continuity

This condition is written as

$$(\rho U)'_x + (\rho V)'_x = 0$$

or

$$\rho (U'_x + V'_y) + U \rho'_x + V \rho'_y = 0$$

with

$$U'_x = U'_{1x} + \left(\frac{\overline{U}}{\xi}\right)'_x y$$

whence

$$V'_y + \frac{\rho'_y}{\rho} V + U'_{1x} + \left(\frac{U}{\xi}\right)'_x y + \frac{\rho'_x}{\rho}\left(U_1 + \frac{U}{\xi} y\right) = 0.$$

Solution of this equation of the first order in V(y) will immediately furnish

$$V(y) = -\frac{1}{\rho}\left[U'_{1x}\int_0^y \rho\, dx + \left(\frac{\overline{U}}{\xi}\right)'_x \int_0^y \rho\, y\, dy + U_1 \int_0^y \rho'_x\, dy + \frac{\overline{U}}{\xi}\int_0^y \rho'_x\, y\, dy\right],$$

to within a constant. This constant is the minimal normal component of the boundary of the sublayer.

[For this we actually have $0 < Y < \epsilon$ and, with $U = \frac{U_1}{\epsilon} y$,

$$V(Y) = -\frac{1}{\rho}\left[U'_{1x}\int_0^Y \rho\, Y\, dY + U_1 \int_0^Y \rho'_x\, Y\, dY\right],$$

so that /154

$$V(\epsilon) = -\frac{1}{\rho}\left[U'_{1x}\int_0^\epsilon \rho\, Y\, dY + U_1 \int_0^\epsilon \rho'_x\, Y\, dY\right]$$

will be extremely small since U'_{1x}, ρ'_x are of the second order of smallness and since ϵ is very weak. Thus, $V(\epsilon) \cong 0$.]

Consequently, V(y) will be of the same order of magnitude as the derivatives with respect to x, i.e., of the second order.

43.2 Navier-Stokes Equations and Elimination of Pressure

Consequently, we put

$$U_{tot} = U + u \quad \text{with} \quad U = U_1 + \frac{\bar{U}}{\xi} y \quad (\text{thus, } U''_{y^2} = 0).$$
$$V_{tot} = V + v.$$

By hypothesis, U_1, ξ, $\bar{U} = U_0 - U_1$ are principal terms, whereas u is small and of the first order while V, v are small and at most of the second order like all derivatives with respect to x. Finally, μ is also small so that the quantities in μU must be retained while those in μu of an order of smallness higher than the second order must be neglected.

In the stationary regime, the Navier-Stokes equations (see Sect.42) reduce to

$$\frac{\partial p}{\partial x} + \rho\left[U\left(U'_x + u'_x\right) + (V + v)\, U'_y\right] \cong \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial g}\right),$$
$$\frac{\partial p}{\partial y} + \rho\left[U\left(V'_x + v'_x\right)\right] \cong 0.$$

(In fact, the terms containing the products of u and μ formed with the derivatives with respect to x can be neglected. This is true also for the terms in Vu'_y, VV'_y, $\mu \frac{\partial V}{\partial g} \cdot \mu \frac{\partial U}{\partial x}$, and $\frac{\partial}{\partial x}\left[\mu \cdot \frac{\partial U}{\partial y}\right]$.)

To complete the condition of continuity, we have

$$(\rho u)'_x + (\rho v)'_y = 0,$$

where $(\rho v)'_y = \rho'_y v + \rho v'_y$ is of the second order and where $(\rho u)'_x = \rho'_x u + \rho u'_x \cong$ $\cong \rho u'_x$ is also of the second order (with $\rho'_x u$ being of the third order).

Let us eliminate p between the two equations by deriving the first with respect to y, the second with respect to x, and then subtracting term by term.

Hence, /155

$$[\rho\{U(U'_x + u'_x) + (V + v)\,U'_y\}]'_y - [\rho\{(U + u)(V'_x + v'_x)\}]'_x$$
$$= \frac{\partial^2}{\partial y^2}(\mu U'_y) = \frac{\bar{U}}{\xi}\mu''_{y^2}$$

(since here $U''_{y^2} = 0$). This relation, equivalent to the "rotation" relation of Part I, is written as follows:

$$U'_y(\rho U'_x + \rho u'_x) + U(\rho U'_x + \rho u'_x)'_y + \{(\rho V)'_y + (\rho v)'_y\}U'_y$$
$$- \{\{\rho U'_x(V'_x + v'_x)\}\} - \rho U(V''_{x^2} + v''_{x^2}) = U'_y \mu''_{y^2}.$$

The term {{ }} is of the fourth order of smallness and thus can be neglected.

Since $\rho u'_x \simeq (\rho u)'_x$, $\rho U'_x \cong (\rho U)'_x - U\rho'_x$ and since $(\rho U)'_x + (\rho V)'_x = 0$, $(\rho u)'_x + (\rho v)'_y = 0$, the following successively remains:

$$U'_y[(\rho U)'_x - U\rho'_x + (\rho u)'_x + (\rho V)'_y + (\rho v)'_y]$$
$$+ U[(\rho U)''_{xy} - (\rho'_x U)'_y + (\rho u)''_{xy} - \rho(V''_{x^2} + v''_{x^2})] = U'_y \mu''_{y^2}$$

and

$$-[U'_y \rho'_x + (\rho'_x U)'_y] U + U[(\rho U)''_{xy} + (\rho u)''_{xy} - \rho(V''_{x^2} + v''_{x^2})] = \frac{\bar{U}}{\xi}\mu''_{y^2}.$$

In addition, we can write

$$(\rho v)'_x \cong \rho v'_x, \qquad (\rho V)''_{x^2} \cong \rho V''_{x^2}, \qquad (\rho v'')_{x^2} \cong \rho v''_{x^2}$$

[the terms in $\rho'_x v$, $\frac{\partial}{\partial x}(\rho'_x v)$, $\frac{\partial}{\partial x}(\rho'_x V)$ can be neglected]. This yields

$$(\rho u)''_{xy} - (\rho v)''_{x^2} + (\rho U)''_{xy} - (\rho V)''_{x^2} = \frac{1}{U}\cdot\frac{\bar{U}}{\xi}\mu''_{y^2} + U'_y\rho'_x + (\rho'_x U)'_y.$$

Let us form

$$(\rho U)''_{xy} = \left[\rho'_y\left(U_1 + \frac{\bar{U}}{\xi}y\right)\right]'_x + \left(\rho\frac{\bar{U}}{\xi}\right)'_x$$
$$= \rho''_{xy}\left(U_1 + \frac{\bar{U}}{\xi}y\right) + \rho'_x\frac{\bar{U}}{\xi} + \rho'_y\left(U'_{1x} + \left(\frac{\bar{U}}{\xi}\right)'_x y\right) + \rho\left(\frac{\bar{U}}{\xi}\right)'_x;$$
$$(\rho'_x U)'_y = \rho''_{xy}\left(U_1 + \frac{\bar{U}}{\xi}y\right) + \rho'_x\frac{\bar{U}}{\xi};$$
$$\rho'_x U'_y = \rho'_x\frac{\bar{U}}{\xi}.$$

Hence, /156

$$(\rho U)''_{xy} - \{\rho'_x U'_y + (\rho'_x U)'_y\} = -\rho'_x\frac{\bar{U}}{\xi} + \rho'_y\left(U'_{1x} + \left(\frac{\bar{U}}{\xi}\right)'_x y\right) + \rho\left(\frac{\bar{U}}{\xi}\right)'_x,$$

leaving finally for the equation of "rotations"

$$(\rho u)''_{xy} - (\rho v)''_{x^2} + \rho\left(\frac{\bar{U}}{\xi}\right)'_x - (\rho V)''_{x^2} = \rho'_x\cdot\frac{\bar{U}}{\xi} - \rho'_y\left\{U'_{1x} + \left(\frac{\bar{U}}{\xi}\right)'_x y\right\} + \frac{\mu''_{y^2}\frac{\bar{U}}{\xi}}{U_1 + \frac{\bar{U}}{\xi}y}.$$

It is also possible to consider a generalized stream function

$$\psi(x, y) = \varphi(y, x)\cdot f(x),$$

such that

$$\psi'_y = (\rho u), \quad \psi'_x = -(\rho v).$$

Thus, for the equation of definition of the function ψ, we will obtain

$$\psi'''_{xy^2} + \psi'''_{x^3} + \rho\left(\frac{\overline{U}}{\xi}\right)'_x - (\rho V)''_{x^2} = \rho'_x \cdot \frac{\overline{U}}{\xi} - \rho'_y \left\{ U'_{1x} + \left(\frac{\overline{U}}{\xi}\right)'_x y \right\} + \frac{\mu''_{y^2} \frac{\overline{U}}{\xi}}{U_1 + \frac{\overline{U}}{\xi} y}.$$

The parallelism of this equation with its homolog obtained in Part I (Sect.3) for the incompressible case is obvious. $\left(\text{At that time, we had } V''_{x^2} = -\left[U'''_{1_{x^3}} y + \left(\frac{\overline{U}}{\xi}\right)'''_{x^3} \frac{y^2}{2}\right].\right)$

The terms on the left-hand side, with variable ρ, are those that had been derived with constant ρ.

The terms on the right-hand side are new; the variations of μ enter here into the analysis of the actual boundary layer.

43.3 Equation of Energy

The conditions of definition of μ, k are those given above, namely

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^{\omega} \qquad \text{i.e.,} \qquad \frac{d\mu}{\mu} = \omega \frac{dT}{T},$$

$$k = \frac{\mu g C_p}{P} \qquad \text{i.e.,} \qquad \frac{dk}{k} = \frac{d\mu}{\mu} = \omega \frac{dT}{T}.$$

It will also be noted that the absolute value of the pressure p is expressed by a large number ($10^4 \times 1.033$ under the so-called "normal" conditions, in MKS units) which will always be the case except at very low pressures. However, for these the medium can no longer be considered as continuous so that the Navier-Stokes equations of continuity are no longer applicable; the present analysis cannot be extended to this case.

Let us return to the Navier-Stokes equations (Sect.43.2) for expressing the pressure gradients: /157

$$\frac{\partial p}{\partial x} = -\rho\,[U\,(U'_x + u'_x) + (V + v)\,U'_y] + \frac{\partial}{\partial y}(\mu\,U'_y).$$

The expression in brackets is of the second order of smallness. With the conventions adopted, the term $\frac{\partial}{\partial y}(\mu U'_y)$ is of a less small order so that

$$\frac{\partial p}{\partial x} \cong \frac{\partial}{\partial y}(\mu U'_y).$$

Here, $\frac{\partial p}{\partial y} = -\rho U(V'_x + v'_x)$ is of the second order.

Hence, for the equation of state $\frac{p}{\rho} = RT$ $\left(\text{where } R = gC_p \cdot \frac{\gamma - 1}{\gamma}\right)$, we obtain by derivation

$$\frac{1}{T}\cdot\frac{\partial T}{\partial x} = \frac{1}{p}\cdot\frac{\partial p}{\partial x} - \frac{1}{\rho}\cdot\frac{\partial \rho}{\partial x} \cong \frac{1}{p}\cdot\frac{\partial}{\partial y}(\mu U'_y) - \frac{1}{\rho}\cdot\frac{\partial \rho}{\partial x} \cong -\frac{1}{\rho}\cdot\frac{\partial \rho}{\partial x}$$

since p is numerically very large.

Thus, $\frac{1}{T} \cdot \frac{\partial T}{\partial x}$ will be of the second order of smallness, just as the other derivatives with respect to x:

$$\frac{1}{T}\cdot\frac{\partial T}{\partial y} = \frac{1}{p}\cdot\frac{\partial p}{\partial y} - \frac{1}{\rho}\cdot\frac{\partial \rho}{\partial y} \cong -\frac{1}{\rho}\cdot\frac{\partial \rho}{\partial y} \quad \text{in principal}$$

Finally, this leads to

$$\frac{d\rho}{\rho} \cong -\frac{dT}{T},$$

i.e.,

$$\rho T \cong \text{const}$$

which is a relation resulting from the stipulated hypotheses.

The above-given equation of energy (Sect.42), in its complete form, will reduce - as is easy to demonstrate - to the terms of the right-hand side forming principal terms

$$U\frac{\partial}{\partial y}(\mu U'_y) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial U}{\partial y}\right)^2 = 0,$$

[all terms on the left-hand side, those in $(V + v)\,\frac{\partial p}{\partial y}$, $\frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right)$, and all terms in μ except those written above will vanish as of the second order at least]. This relation shows that, in principal, T is a function of U(x, y) and of y:

$$T = T(U, y).$$

Substituting μ by $\mu = \frac{P}{gC_p}\,k$, we obtain /158

$$\frac{\partial}{\partial y}\left[k\frac{\partial T}{\partial y}\right]+\frac{P}{gC_p}\left[U\frac{\partial}{\partial y}(kU'_y)+kU'^2_y\right]=0.$$

Here,

$$\frac{\partial T}{\partial y}=\left(\frac{\partial T}{\partial y}\right)_{U=\text{const}}+\frac{\partial T}{\partial U}\cdot\frac{\partial U}{\partial y},$$

whence

$$\frac{\partial}{\partial y}\left[k\frac{\partial T}{\partial y}\right]=\frac{\partial}{\partial y}\left[k\left\{\left(\frac{\partial T}{\partial y}\right)_{U=\text{const}}+\frac{\partial T}{\partial U}U'_y\right\}\right]$$
$$=k\left[\left(\frac{\partial^2 T}{\partial y^2}\right)_{U=\text{const}}+\frac{\partial^2 T}{\partial U^2}U'^2_y+\frac{\partial T}{\partial U}U''_{y^2}\right]+\frac{\partial k}{\partial y}\left[\left(\frac{\partial T}{\partial y}\right)_{U=\text{const}}+\frac{\partial T}{\partial U}\cdot U'_y\right];$$

$$\frac{\partial}{\partial y}[kU'_y]=\frac{\partial k}{\partial y}U'_y+kU''_{y^2}.$$

This leaves, for the equation of energy,

$$\left[k\frac{\partial^2 T}{\partial y^2}+\frac{\partial k}{\partial y}\cdot\frac{\partial T}{\partial y}\right]+\left[kU'^2_y\left(\frac{\partial^2 T}{\partial U^2}+\frac{P}{gC_p}\right)+\left(U'_y\frac{\partial k}{\partial y}+kU''_{y^2}\right)\left(\frac{\partial T}{\partial U}+\frac{P}{gC_p}U\right)\right]=0.$$

The second expression in brackets has the following solution:

$$\frac{\partial T}{\partial U}=-\frac{P}{gC_p}U$$

or

$$T=-\frac{P}{2gC_p}U^2+C_1 \quad \text{(constant with respect to U).}$$

In the first bracket, in view of the fact that $\frac{1}{k}\cdot\frac{\partial k}{\partial y}=\frac{\omega}{T}\cdot\frac{\partial T}{\partial y}$, the following remains:

$$\frac{\partial^2 T}{\partial y^2}+\frac{\omega}{T}\left(\frac{\partial T}{\partial y}\right)^2=0$$

which is also written as

$$T^{m-1}\cdot\frac{\partial^2 T}{\partial y^2}+\omega T^{m-2}\left(\frac{\partial T}{\partial y}\right)^2=0.$$

However,

$$(T^m)'_y=mT^{m-1}\cdot T'_y, \qquad (T^m)''_{y^2}=m\left[(m-1)T^{m-2}T'^2_y+T^{m-1}\cdot T''_{y^2}\right].$$

Let us here put $m-1=\omega$ or $m=\omega+1$. This yields

$$[T^{(\omega+1)}]''_{y^2}=0,$$

i.e., /159

$$[T^{(\omega+1)}]'_y = B_1 \quad \text{and} \quad T^{(\omega+1)} = B_1 y + B_2,$$

where B_1 and B_2 are constants with respect to y (and U).

Thus, the general expression of T is

$$T = -\frac{P}{2gC_p} U^2 + [B_1 y + B_2]^{\frac{1}{1+\omega}};$$

$$\frac{\partial T}{\partial y} = -\frac{P}{gC_p} U \cdot U'_y + \frac{B_1}{(1+\omega)[B_1 y + B_2]^{\frac{\omega}{1+\omega}}}.$$

It should be noted here that all reasonings and all approximations made with respect to the equation of energy apply just as well to the sublayer, when setting $U = \frac{U_1}{\epsilon} Y$ for $0 < Y < \epsilon$, as to the actual boundary layer, with

$$0 < y < \xi, \quad U = U_1 + \frac{U}{\xi} y.$$

Since we retained the terms in U''_{y^2} in all of the expansions, they are applicable also to the zone of contact between the two layers (at the interior of each of these, $U''_{y^2} = 0$).

The obtained solution T(U, y), which is an approximate solution, is valid for the entire turbulent boundary layer.

Two particular cases can be specifically investigated:

First case:

Adiabatic wall $\left(\frac{\partial T}{\partial y}\right)_w = 0$.

(Here, w is the subscript referring to the state at the wall where U = 0, Y = = y = 0).

Outside the boundary layer, we have

$$T(\epsilon + \xi) = T_0 \text{ imposed}$$

with

$$U(\epsilon + \xi) = U_0.$$

Hence,

$$B_1 = 0, \quad T_0 = -\frac{P}{2gC_p} U_0^2 + [B_1(\epsilon + \xi) + B_2]^{\frac{1}{1+\omega}}.$$

From this it follows that

$$B_2 = \left[T_0 + \frac{P}{2\,gC_p} U_0^2\right]^{1+\omega},$$

so that

$$T = -\frac{P}{2\,gC_p} U^2 + T_0 + \frac{P}{2\,gC_p} U_0^2 = T_0 - \frac{P}{2\,gC_p} (U^2 - U_0^2).$$

Here, the "recovery factor" assumes the value P, with* /160

$$T_W = T_0 + \frac{P}{2\,gC_p} U_0^2.$$

Second case:

The wall is at an imposed temperature T_w since T_0 is also imposed. Then,

$$T_W = B_2^{\frac{1}{1+\omega}} \qquad \text{i. e.,} \qquad B_2 = T_W^{(1+\omega)},$$

$$T_0 = -\frac{P}{2\,gC_p} U_0^2 + [B_1(\varepsilon + \xi) + T_W^{(1+\omega)}]^{\frac{1}{1+\omega}}.$$

Hence,

$$B_1 = \frac{1}{(\varepsilon + \xi)} \left[\left(T_0 + \frac{P}{2\,gC_p} U_0^2\right)^{(1+\omega)} - T_W^{(1+\omega)}\right]$$

and

$$T(Y) = -\frac{P}{2\,gC_p} U^2(Y) + \left[\frac{Y}{\varepsilon + \xi} \left\{\left(T_0 + \frac{P}{2\,gC_p} U_0^2\right)^{(1+\omega)} - T_W^{(1+\omega)}\right\} + T_W^{(1+\omega)}\right]^{\frac{1}{1+\omega}}.$$

The heat exchange between the wall and the fluid is then characterized by

$$\frac{dQ}{dt \cdot dx} = k \left(\frac{\partial T}{\partial Y}\right)_W \qquad \text{where} \qquad k = k_0 \left(\frac{T_W}{T_0}\right)^{\omega}.$$

We obtain

$$\frac{dQ}{dt \cdot dx} = \frac{k_0}{1+\omega} \left(\frac{T_W}{T_0}\right)^{\omega} \cdot \frac{1}{(\varepsilon + \xi)} \left[\left(T_0 + \frac{P}{2\,gC_p} U_0^2\right)^{(1+\omega)} - T_W^{(1+\omega)}\right] \frac{1}{T_W^{\omega}}$$

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* Two different assumptions can be made with respect to this result:
The presented method is only approximate. A second approximation which contains the terms of the second order, neglected in the equation of energy, would permit greater accuracy.
The method does not incorporate the effects of the nonstationary perturbation components of which we know that they are always present in the turbulent state.

$$= \frac{k_0}{(1+\omega)T_0^{\omega}} \cdot \frac{1}{(\epsilon+\xi)} \left[\left(T_0 + \frac{P}{2gC_p} U_0^2 \right)^{(1+\omega)} - T_w^{(1+\omega)} \right].$$

43.4 Investigation of the Velocity Field

No matter what case is considered, this study indicates that the temperature field in the turbulent layer, in first approximation, is connected with the following quantities:

$(\epsilon + \xi)$, total thickness of the boundary layer;

$U = \frac{U_1}{\epsilon} Y$ in the sublayer, and $U = U_1 + \frac{U}{\xi} y$ in the actual boundary layer;

specific mass ρ, such that $\rho T \cong \rho_0 T_0$. Similarly, $\mu = \mu_0 \left(\frac{T}{T_0}\right)^{\omega}$.

When studying the problem in y where U_1, $\overline{U} = U_0 - U_1$, ξ (and ϵ) are assumed as known*, the term T(Y) will also be known from the instant at which both boundary conditions are fixed. /161

Then, $\rho(Y)$ and $\mu = \mu_0 \left(\frac{T}{T_0}\right)^{\omega}$ will be known. This makes it possible, at least numerically, to calculate V(y) (see its expression in Sect.43.1).

In the equation in ψ (Sect.43.2), all terms other than those in ψ can be calculated. If, specifically, the development $\rho(y)$ and $\left(\frac{\mu(y)}{U_1 + \left(\frac{U}{\xi}\right)_y}\right)$ is expressed in the form of algebraic expansions in powers of y such that $\rho = \rho_1 + \rho_2 y + \rho_3 y^2 \ldots$ (where the ρ_n are a function only of x), then the solution will be of the same type as those given in Part I (Sects.3.1, 3.2, etc.). This solution will furnish a decomposition parallel to that already obtained in the incompressible regime (terms in sin $c_n y$ and in powers of y; see Sect.5.1).

As before, it then becomes possible to use the equation previously derived for the base of the actual boundary layer, together with the complementary terms, for studying the problem in x.

When making U'_{1_x}, $\overline{U}'_x$ tend to zero, it should be possible to obtain the asymptotic solution and to write the equation in $\overline{\xi}(x)$ homologous to eq.(Ib) in Sections 17 and 18. Similarly, by means of the equation of momentum loss homologous to eq.(IIIb) in Sections 11.1 and 11.2, a relation for defining the asymptotic friction C'_f will be obtained. Here, the form of the corresponding solutions can be anticipated.

Our purpose in this Chapter was merely to demonstrate that, with the aid of

* Since ϵ, no doubt, is still extremely small no great error will be committed in neglecting it.

the basic schemes and the approximations used for the incompressible regime, the compressible problem can be attacked in a simple and detailed manner, even in the presence of heat exchange.

This finally relates to the following facts:

possibility to write the equation in ψ as a complementary generalized stream function (with respect to the complementary component u) in a form very similar to that given for the incompressible regime, yielding a solution of an analogous form if $T(y)$ and, consequently, also $\mu(y)$, $\rho(y)$ can be suitably expressed;

possibility to find, independently of the solution of the equation in ψ, the law of evolution $T(y)$ and, by the approximate relation obtained from the adopted approximations $\rho T \cong$ const, to define $\rho(y)$, with both propositions resulting from a study of the equation of energy.

CONCLUSIONS

The theory presented here leads to a set of conclusions whose major portion merely reflects already known experimental phenomena.

In terms of these conclusions, the mechanism of flow in the boundary layer on a plane plate would be characterized by the following properties:

1) At very low Reynolds numbers, the laminar local friction coefficient is larger than that corresponding to the "turbulent" flow so that the laminar state is at maximal entropy. Thus, this is the state that becomes established here; it is rotational, is at constant pressure, and has weak trajectory curvatures.

If the exterior flow is perfectly calm and the wall is perfectly smooth, the laminar state will persist up to the largest Reynolds numbers.

2) The external perturbations induce a reaction in the boundary layer (essentially in v')* of the same frequency as the excitation, a process that continues up to a certain critical Reynolds number linked to the excitation wavelength:

$$\frac{1}{2}\mathcal{R}_c = 0.0017\,\lambda^2\,\frac{U_0^2}{\nu^2}.$$

From the segment $\frac{x_c}{2}$ defined in this manner up to the segment x_c $\left(\Re_c = 0.0034\,\lambda^2\,\frac{U_0^2}{\nu^2}\right)$, a series of secondary nonstationary responses (in u'_{II}) of increasing frequency and nonzero time averages are generated* such that the sum of these averages, at each level y_1 of the segment x_c, makes up the difference existing between the "stationary turbulent" and laminar fields $U(y)$, with the "stationary turbulent" field being a solution of the Navier-Stokes equations at nonconstant pressure.

The domain of the segments x such that $\frac{\Re_c}{2} < \Re < \Re_c$ is that of the transition along which nonstationary perturbations u'_{II} develop as a secondary response to the external excitation; these perturbations are maintained as such beyond $\Re_c$, with a well-defined frequency and elongation spectrum, to form a so-called "turbulent" state exactly because of the presence of these nonstationary components that become permanently established.

(The frequencies attain kilocycles or even tens of kilocycles.) The situation thus is as follows:

The segment x_c is the last segment in which the laminar state can persist

* Here, v' is the normal nonstationary perturbation velocity component while u' is the tangential component.

since, in the presence of an external perturbation, pressure gradients will develop there that are not compatible with the invariance of pressure characteristic for the laminar state.
The secondary perturbations (in u'_{II}) cannot propagate against the stream in the harmonic state (possibility of unpredicted flow separation) and cannot be generated upstream of $\frac{x_c}{2}$ if the critical segment corresponding to the preceding statement is x_c.

3) From the theory it is found that the mean turbulent field, at the same value of $\Re$, is characterized with respect to the laminar field by considerable increases in the boundary layer thickness and in the friction coefficient.

This is correlated with the configuration of the field U(y) such that, very close to the wall, a sublayer with a strong gradient U'_y appears, superposed by a layer in which the evolution of U is much slower than in the laminar layer (which is slower the greater the thickness of the layer). It is in this sublayer that the strongest rotations and the fundamental viscosity effects are located. The thickness of the layer is a few tenths of a millimeter. The layer is bounded by a line where the velocity (U_1) assumes a well-defined value (about $0.45\ U_0$).

The approximate theory indicates that the actual boundary layer has a finite thickness; however, the hypotheses of the theory permit no formal conclusions in this respect.

4) The turbulent states are such that, irrespective of the value of $\Re_c$, condensation of the curves of the coefficient of friction into a single curve $\left(\text{slowly decreasing; in the main like } \sqrt[4]{\frac{\beta}{\Re}}\right)$ takes place for $\Re > \Re_c$, defining an asymptotic state of the boundary layer spectrum.

The curve $C_f^*(\Re)$ intersects that of the laminar (local) friction coefficient for a low value of the Reynolds number $\Re_0$ $\left(\text{usually, very small with respect to } \frac{1}{2}\Re_c\right)$.

Thus, starting from $\Re_0$, the turbulent state is that of maximal entropy. Nevertheless, if the turbulent state originates in $\Re_c$, it will not continue up to $\Re_0$ but only to $\frac{1}{2}\Re_c$, since the secondary perturbations interior to u'_{II} cannot exist upstream of $\frac{1}{2}\Re_c$.

5) The theory permits a detailed calculation of the velocity fields of a turbulent boundary layer on a plane plate, from their evolution in the transition, from the localization of this transition, and from the development of the local and total friction coefficients.

A comparison of the results of these calculations with practical experiments shows agreement with the experimental results, almost as satisfactory as that obtained - in the laminar regime - from the Blasius theory.

6) There exists a possibility of maintaining the laminar state if the wall, instead of being rigid, is constituted of an elastic membrane stretched over a constant-pressure layer, with the tension linked to the pulsation of the external perturbation which, in the usual case, induces passage to the turbulent state. /164

7) It is possible to proceed from here to a study of the compressible case with heat exchange, using the same basic schemes and the same approximations, as well as to attack the problem by means of a solution parallel to that used for the incompressible case.

APPENDICES

APPENDIX I

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EXPANSION OF THE BLASIUS LAW IN THREE-TERM EXPONENTIALS

Let $f'_\eta(\eta)$ be the Blasius function such that $f'_\eta = 1 - \Phi(\eta)$.

For $\eta = y\sqrt{\frac{U_0}{\nu X}} = 0$, it is necessary that $\Phi = 1$ $\Phi'_\eta = \Phi'_0$ is given.

Let us put

$$\Phi(\eta) = a_1 e^{-\bar{\alpha}_i(i-\varepsilon)\eta} + a_2 e^{--\bar{\alpha}_i(1+\varepsilon)\eta} + a_2 e^{-\bar{\alpha}_i\eta}$$

and also

$$x = \bar{\alpha}_i \eta \quad (\text{whence } \bar{x}'_\eta = \bar{\alpha}_i).$$

Then,

$$\Phi = a_1 e^{-(1-\varepsilon)x} + a_2 e^{-(1+\varepsilon)x} + a_2 e^{-x} = e^{-x}\left[a_1 e^{\varepsilon x} + a_2 e^{-\varepsilon x} + a_3\right].$$

For x = 0, the term $\Phi(0) = 1$ leads to

$$a_1 + a_2 + a_3 = 1.$$

$$\Phi'_\eta = \Phi'_x \cdot x'_\eta = \left[-a_1(1-\varepsilon)e^{-(1-\varepsilon)x} - a_2(1+\varepsilon)e^{-(1+\varepsilon)x} - a_3 e^{-x}\right]\bar{\alpha}_i,$$

whence

$$\Phi'_\eta(0) = \Phi'_0 = -\bar{\alpha}_i\left[a_1(1-\varepsilon) + a_2(1+\varepsilon) + a_3\right]$$
$$= -\bar{\alpha}_i\left[a_1 + a_2 + a_3 - \varepsilon(a_1 - a_2)\right] = -\bar{\alpha}_i\left[1 - \varepsilon(a_1 - a_2)\right],$$

which yields

$$\bar{\alpha}_i \varepsilon (a_1 - a_2) - \bar{\alpha}_i = \Phi'_0,$$

i.e.,

$$a_1 - a_2 = \frac{1}{\varepsilon}\left[\frac{\Phi'_0}{\bar{\alpha}_i} + 1\right].$$

Then,

$$\Phi''_{\eta^2} = x'^2_\eta \cdot \Phi''_{x^2} = \bar{\alpha}_i^2\left[a_1(1-\varepsilon)^2 e^{-(1-\varepsilon)x} + a_2(1+\varepsilon)^2 e^{-(1+\varepsilon)x} + a_3 e^{-x}\right]$$
$$= \bar{\alpha}_i^2 e^{-x}\left[a_1(1-\varepsilon)^2 e^{\varepsilon x} + a_2(1+\varepsilon)^2 e^{-\varepsilon x} + a_3\right]$$
$$\cong \bar{\alpha}_i^2 e^{-x}\left[a_1 e^{\varepsilon x} + a_2 e^{-\varepsilon x} + a_3 - 2\varepsilon\left(a_1 e^{\varepsilon x} - a_2 e^{-\varepsilon x}\right)\right].$$

Consequently, /168

$$\frac{\Phi''^2_\eta}{\Phi} = \bar{\alpha}_i^2\left[1 - 2\,\varepsilon \cdot \frac{a_1 e^{+\varepsilon x} - a_2 e^{-\varepsilon x}}{a_1 e^{+\varepsilon x} + a_2 e^{-\varepsilon x} + a_3}\right] \cong \bar{\alpha}_i^2\left[1 - 2\,\varepsilon \cdot \frac{a_1 - a_2 - \varepsilon x (a_1 - a_2)}{a_1 + a_2 + a_3 + \varepsilon x (a_1 - a_2)}\right]$$

$$= \bar{\alpha}_i^2\left[1 - 2\,\varepsilon \frac{a_1 - a_2}{a_1 + a_2 + a_3}\left\{1 + \varepsilon x\left(\frac{a_1 + a_2}{a_1 - a_2} - \frac{a_1 - a_2}{a_1 + a_2}\right)\right\}\right].$$

Since

$$a_1 + a_2 + a_3 = 1, \qquad a_1 - a_2 = \frac{1}{2}\left(\frac{\Phi'_0}{\bar{\alpha}_i} + 1\right),$$

it follows that

$$\frac{\Phi''^2_\eta}{\Phi} \cong \bar{\alpha}_i^2\left[1 - 2\left(\frac{\Phi'_0}{\bar{\alpha}_i} + 1\right)\left\{1 - \varepsilon x\left(\frac{\varepsilon (a_1 + a_3)}{\frac{\Phi'_0}{\bar{\alpha}_i} + 1} - \frac{\frac{\Phi'_0}{\bar{\alpha}_i} + 1}{\varepsilon (a_1 + a_3)}\right)\right\}\right]$$

$$\cong \alpha_i^2\left[1 - 2\left(\frac{\Phi'_0}{\bar{\alpha}_i} + 1\right) - 2x\left(\frac{\Phi'_0}{\bar{\alpha}_i} + 1\right)^2\right]$$

not retaining the term which can be neglected in ε^2. Then $\frac{\Phi''^2_\eta}{\Phi}$ = const if $\frac{\Phi'_0}{\bar{\alpha}_1} + 1 = 0$, i.e., if $\bar{\alpha}_1 = -\Phi'_0$ and $a_1 = a_2$. This leaves

$$\frac{\Phi''^2_i}{\Phi} = \bar{\alpha}_i^2$$

so that we have

$$\Phi = e^{-\bar{\alpha}_i \eta}\,[2\,a_1 \cosh \varepsilon\,\bar{\alpha}_i\,\eta + a_3].$$

Blasius field at $U'_{0_x} = 0$:

$$\Phi = e^{-0.375\eta}\,[73.5 - 72.5 \cosh 0.037^5 \cdot \eta],\ (\varepsilon = 0.1).$$

η	0	1	2	3	4	5
Φ	1	0.645	0.375	0.172	0.050	~ 0
$1 - \Phi$	0	0.355	0.625	0.828	0.950	~ 1

APPENDIX II

CASE OF NONZERO VELOCITY GRADIENT U'_{0_x} OF THE EXTERIOR FLOW

The velocity spectrum of a laminar layer of a plane plate, inclined with respect to the stream for which $U'_{0_x} \neq 0$, is that given by Blasius and modified by Polhausen:

$$\frac{U}{U_0} = f'_\eta(\eta) + \frac{A}{6}\varkappa(1-\varkappa)^3,$$

where $\varkappa = \frac{y}{\delta}$ and where f'_η is the Blasius function $\eta = y\sqrt{\frac{U_0}{\nu x}}$ and where, finally, $A = \frac{\delta^2}{\nu} U'_{0_x}$.

The change in the configuration of the field $\frac{U(y)}{U_0}$ is shown in Diagram VII (Fig.35). It is also possible to use an image function of the same form as that given above, at least for sufficiently small $\frac{U'_{0x}}{U_0}$, the only case which will be investigated here:

$$U = U_0\left[1 - \sum_i a_i e^{-\alpha_i y\sqrt{\frac{U_0}{\nu x}}}\right].$$

The coefficients a_i, α_i naturally will undergo modifications. The function U'_x will become

$$U'_x = \Big[-U_0\sum_i a_i \alpha_i y \frac{1}{2x}\sqrt{\frac{U_0}{\nu x}} + U'_{0_x}\Big\{1 - \sum_i a_i \times \left(1 - \alpha_i y \frac{1}{2U_0}\sqrt{\frac{U_0}{\nu x}}\right)\Big\}\Big] e^{-\alpha_i y\sqrt{\frac{U_0}{\nu x}}}.$$

Returning to the statements in Chapter I, it is easy to demonstrate that the Navier-Stokes equations for defining the stream function ψ of the complementary velocities u, v contains U'_x only at the second infinitesimal order if U'_{0_x} is small, as we are assuming here. Thus, nothing is changed in solving the problem in Y, except the numerical values of the coefficients Φ_{n_j} which are here determined by the Polhausen field.

This field is distinguished from the Blasius field by using the notation $\bar{\alpha}_i^*$ which will be substituted for $\bar{\alpha}_i$ relative to the Blasius field. According to the very configuration of these fields, we have

$$\bar{\alpha}_i^* > \bar{\alpha}_i \ \text{ if } \ U'_{0_x} > 0, \qquad \bar{\alpha}_i^* < \bar{\alpha}_i \ \text{ if } \ U'_{0_x} < 0.$$

In the presence of an external perturbation of pulsation αU_0, the boundary condition of the laminar domain will be /170

$$x < x_c^* \text{ where } x_c^* \text{ such that } \alpha \leqslant \bar{\alpha}_i^* \sqrt{\frac{U_0}{\nu x_c^*}}, \qquad x_c^* = \frac{\bar{\alpha}_i^{*2}}{\alpha^2} \cdot \frac{U_0}{\nu}.$$

instead of the relative $x_c = \dfrac{\bar{\alpha}_i^2}{\alpha^2} \cdot \dfrac{U_0}{\nu}$, for the same local value U_0 at conditions corresponding to $U'_{0_x} = 0$.

It can be proved directly that

$$x_c^{**} > x_c \text{ if } U'_{0_x} > 0, \qquad x_c^* < x_c \text{ if } U'_{0_x} < 0.$$

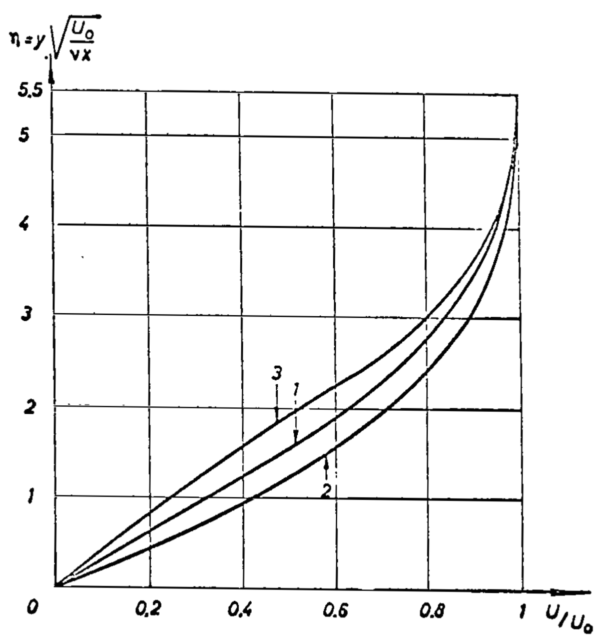

Fig.35

Diagram VII (Fig.35):

1) Blasius field $U'_{0_x} = 0$.

2) Polhausen field $\dfrac{U'_{0_x}}{U_0} > 0$.

3) Polhausen field $\dfrac{U''_{0_x}}{U_0} < 0$.

APPENDIX III

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PERTURBATION INTERIOR TO THE BOUNDARY LAYER; SECOND APPROXIMATION

Let us return to the equation of space relative to a function f_1:

$$f'''_{x^3} + f'_x\left(k^2 - \alpha_i^2\,\frac{U_0}{\nu x}\right) = 0. \tag{1}$$

We again operate step by step, assuming that at a point $x = x_0$ the solution $f(x_0)$, $f'_x(x_0)$ is known and then calculating, for $x = x_0 + \Delta x$ $\left(\frac{\Delta x}{x_0}\text{ being small}\right)$, the new values of $f(x)$, $f'_x(x)$. Then eq.(1) is written as follows:

$$f'''_{x^3} + f'_x\left[k^2 - \alpha_i^2\,\frac{U_0}{\nu x_0}\left\{1 - \frac{\Delta x}{x_0} + \ldots\right\}\right] = 0. \tag{1a}$$

The first approximation furnished the solution corresponding to

$$f'''_{x^3} + f'_x\left[k^2 - \alpha_i^2\,\frac{U_0}{\nu x_0}\right] = 0,$$

namely, since β is imaginary here*,

$$f'_x = \mathring{i}\sqrt{k^2 - \alpha_i^2\,\frac{U_0}{\nu x_0}}\left[f_1\, e^{\mathring{i}\sqrt{k^2-\alpha_i^2\frac{U_0}{\nu x_0}}\cdot\Delta x} - f_2\, e^{-\mathring{i}\sqrt{k^2-\alpha_i^2\frac{U_0}{\nu x_0}}\cdot\Delta x}\right].$$

The solution of eq.(1a) will produce a modification $\delta f(x)$ of this first approximation. We will seek the value by linearization, from which the following condition for $\delta f'_x$ is obtained:

$$\delta f'''_{x^3} + \delta f'_x\left(k^2 - \alpha_i^2\,\frac{U_0}{\nu x_0}\right) + \left[f'_x\,\alpha_i^2\,\frac{U_0}{\nu x_0}\left(\frac{\Delta x}{x_0} - \frac{\Delta x^2}{x_0^2}\ldots\right)\right] = 0. \tag{2}$$

Since /172

$$f'_x = \mathring{i}\sqrt{k^2 - \alpha_i^2\,\frac{U_0}{\nu x_0}}\left[f_1 - f_2 + \mathring{i}\sqrt{k^2 - \alpha_i^2\,\frac{U_0}{\nu x_0}}\,(f_1 + f_2)\,\Delta x\ldots\right],$$

* A real β $|\mu'|$, increasing indefinitely, could refer to the case of flow separation. This is not what we propose here.

the solution $\delta f'_x$ has the form

$$\delta f'_x = \overset{\circ}{i}\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}$$

$$\times \left[\delta f_1 e^{\overset{\circ}{i}\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}\Delta x} - \delta f_2 e^{-\overset{\circ}{i}\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}\Delta x} \right] + b + 2c\Delta x + 3e\Delta x^2 \ldots.$$

The expansion of $\delta f'_x$ must be continued to terms in Δx of a power higher than unity since, because of the fact that the first approximation here satisfies $f'''_{x_3} + f'_x \left(k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}\right) = 0$, the term f'_x will contain Δx as factor since $f_1 - f_2 = 0$ will vanish (to satisfy the boundary conditions).

The term on the right-hand side in brackets, with respect to the equation in $\delta f'_x$, will contain Δx^2 as factor.

The expression in brackets corresponding to the solution with zero right-hand side identifies this solution with that of the first approximation and thus can be incorporated into it ($\delta f_1 = \delta f_2 = 0$).

For determining the coefficients b, c, e, the classical procedure of identifying all terms of like power in Δx is followed in the general equation (2) which now is written in the form

$$6e + (3e\Delta x^2 + 2c\Delta x + b)\left(k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}\right) + \alpha_i^2 \frac{U_0}{\nu x_0}\left(\frac{\Delta x}{x_0} - \frac{\Delta x^2}{x_0^2}\ldots\right)\overset{\circ}{i}\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}$$

$$\times \left\{ f_1 - f_2 + \overset{\circ}{i}\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}\,(f_1 + f_2)\,\Delta x - \ldots \right\} = 0,$$

whence

constant term:

$$6e + b\left(k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}\right) = 0;$$

term in Δx:

$$2c\left(k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}\right) + \alpha_i^2 \frac{U_0}{\nu x_0}\,\overset{\circ}{i}\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}} \cdot \frac{1}{x_0}(f_1 - f_2) = 0;$$

term in Δx^2:

$$3e\left(k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}\right) + \alpha_i^2 \frac{U_0}{\nu x_0}\,\overset{\circ}{i}\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}$$

$$\times \left[\frac{1}{x_0}\overset{\circ}{i}\sqrt{k^2 - \alpha_i^2 \frac{U_0}{\nu x_0}}\,(f_1 + f_2) - \frac{1}{x_0}(f_1 - f_2)\right] = 0.$$

Hence, /173

$$3\,e = \frac{1}{x_0}\,\alpha_l^2\,\frac{U_0}{\nu x_0}\left[(f_1 + f_2) + \frac{\overset{\circ}{i}}{x_0}\cdot\frac{f_1 - f_2}{\sqrt{k^2 - \alpha_l^2\,\frac{U_0}{\nu x_0}}}\right],$$

$$2\,c = -\,\alpha_l^2\,\frac{U_0}{\nu x_0}\cdot\frac{\overset{\circ}{i}}{x_0}\cdot\frac{f_1 - f_2}{\sqrt{k^2 - \alpha_l^2\,\frac{U_0}{\nu x_0}}},$$

$$b = -\,\frac{6\,e}{k^2 - \alpha_l^2\,\frac{U_0}{\nu x_0}} = -\,\frac{2}{x_0}\cdot\frac{\alpha_l^2\,\frac{U_0}{\nu x_0}}{k^2 - \alpha_l^2\,\frac{U_0}{\nu x_0}}\left[f_1 + f_2 + \frac{\overset{\circ}{i}}{x_0}\cdot\frac{f_1 - f_2}{k^2 - \alpha_l^2\,\frac{U_0}{\nu x_0}}\right].$$

It can be verified that the related facts, concerning the propagations and limitations stipulated by the first approximation of f(x, t), are encountered again in the second approximation.

Let us return first to the time condition:

$$k^2\,(f'_t + U_0\,f'_x) + f'''_{x^2 t} + U_0\,f'''_{x^3} = \nu\,[k^4 f + 2\,k^2 f''_{x^2} + f''''_{x^4}],$$

i.e.,

$$\frac{\partial}{\partial t}\,[f''_{x^2} + k^2 f] + U_0\,[f'''_{x^3} + k^2 f'_x] - \nu\,[k^4 f + 2\,k^2 f''_{x^2} + f''''_{x^4}] = 0.$$

Here,

$$f = f_1\,e^{\beta\Delta x} + f_2\,e^{-\beta\Delta x} + (f_1 + f_2)\cdot\frac{2}{x_0}\cdot\frac{1}{\beta^2}\cdot\alpha_l^2\,\frac{U_0}{\nu x_0}\,\Delta x + \ldots + f_3,$$

with

$$\beta = \sqrt{\alpha_l^2\,\frac{U_0}{\nu x_0} - k^2}, \qquad f_1, f_2 \text{ and } f_3 \text{ functions of time.}$$

Since Δx is small, we can write f in the following form:

$$f = f_1\,e^{\left(\beta + \frac{2}{x_0}\cdot\frac{\alpha_l^2}{\beta^2}\cdot\frac{U_0}{\nu x_0}\right)\Delta x} + f_2\,e^{-\left(\beta - \frac{2}{x_0}\cdot\frac{\alpha_l^2}{\beta^2}\cdot\frac{\nu x_0}{U_0}\right)\Delta x} + f_3 = f_1\,e^{(\beta+\gamma)\Delta x} + f_2\,e^{-(\beta-\gamma)\Delta x} + f_3,$$

by putting

$$\gamma = \frac{2}{x_0}\cdot\frac{\alpha_l^2}{\beta^2}\cdot\frac{U_0}{\nu x_0}.$$

Let us enter f, made explicit in this manner, into the equation of time. /174
For the term having $e^{(\beta+\gamma)\Delta x}$ as factor, we then obtain

$$f'_{1_t}\{(\beta+\gamma)^2+k^2\}$$
$$+f_1[U_0(\beta+\gamma)\{(\beta+\gamma)^2+k^2\}-\nu\{k^4+2k^2(\beta+\gamma)^2+(\beta+\gamma)^4\}]=0,$$

whence

$$f_1=f_{1_0}{}^{-U_0(\beta+\gamma)\left[1-\frac{\nu}{U_0}\cdot\frac{(\beta+\gamma)^2+k^2}{\beta+\gamma}\right]\Delta t}\cong f_{1_0}e^{-U_0(\beta+\gamma)\Delta t},$$

where f_{1_0} is a constant.

Similarly,

$$f_2=f_{2_0}e^{U_0(\beta-\gamma)\left[1+\frac{\nu}{U_0}\cdot\frac{(\beta-\gamma)^2+k^2}{\beta-\gamma}\right]\Delta t}\cong f_{2_0}e^{U_0(\beta-\gamma)\Delta t}.$$

Finally,

$$k f'_{3_t}-\nu k^4 f_3=0 \quad \text{yields} \quad f_3\cong f_{3_0}e^{\nu k^2 t}\cong \text{const.}$$

From this follow the general expressions of f:

$$f=f_{1_0}e^{(\beta+\gamma)\left[\Delta x-U_0\Delta t\cdot\left\{1-\frac{\nu}{U_0}\cdot\frac{(\beta+\gamma)^2+k^2}{\beta+\gamma}\right\}\right]}$$
$$+f_{2_0}e^{-(\beta-\gamma)\left[\Delta x-U_0\Delta t\left\{1+\frac{\nu}{U_0}\cdot\frac{(\beta+\gamma)^2+k^2}{\beta-\gamma}\right\}\right]}+f_{3_0}e^{\nu k^2\Delta t},$$

which cause the following velocities of propagation to appear:

$$\mathfrak{a}_1=U_0\left\{1-\frac{\nu}{U_0}\cdot\frac{(\beta+\gamma)^2+k^2}{\beta+\gamma}\right\},\qquad \mathfrak{a}_2=U_0\left\{1+\frac{\nu}{U_0}\cdot\frac{(\beta-\gamma^2)+k^2}{\beta-\gamma}\right\}.$$

The exponential terms reduce to constants for f_1, if $\beta+\gamma\cong 0$ and for f_2, if $\beta\simeq\gamma\cong 0$. These are written as

$$\sqrt{\alpha_i^2\frac{U_0}{\nu x}-k^2}=\mp\frac{2}{x}\cdot\frac{\alpha_i^2\frac{U_0}{\nu x}}{\alpha_i^2\frac{U_0}{\nu x}-k^2},$$

i.e.,

$$\frac{4}{x^2}\cdot\left(\alpha_i^2\frac{U_0}{\nu x}\right)^2=\left(\alpha_i^2\frac{U_0}{\nu x}-k^2\right)^3.$$

For a given $\frac{k^2}{\alpha^2}$, for example equal to 2, we obtain

$$\left(\alpha^2 \frac{x_c}{x} - 2\alpha^2\right)^3 = \frac{4}{x^2}\alpha^4 \cdot \frac{x_c^2}{x^2},$$

i.e., /175

$$\left(\frac{x^4}{x_c^4} - 1.5\frac{x^3}{x_c^3} + 0.75\frac{x^2}{x_c^2} - 0.125 \cdot \frac{x}{x_c}\right) - \frac{1}{2} \cdot \frac{1}{\alpha^2 x_c^2} = 0.$$

Since

$$\alpha^2 = \alpha_i^2 \frac{U_0}{\nu x_c}, \qquad \alpha^2 x_c^2 = \alpha_i^2 \frac{U_0}{\nu} x_c = \alpha_i^2 \mathcal{R}_c,$$

the term $\frac{1}{2\alpha^2 x_c^2} = \frac{1}{2\alpha_i^2 \mathfrak{R}_c}$ is extremely small. The expression in parentheses has $\frac{x}{x_c} = \frac{1}{2}$ as root. It is easy to demonstrate that the perturbation ceases being harmonic in the direct vicinity of $\frac{x_c}{2}$, as had been observed for the first approximation in the same case $k^2 = 2\alpha^2$, and that it must vanish in this immediate vicinity $\left(\text{as well as upstream of } \frac{x_c}{2}\right)$.

So far as the function g is concerned, the space relation is written as follows:

$$g'''_{x^3} - \alpha_i^2 \frac{U_0}{\nu x} g'_x = 0.$$

Introducing, as done before for f, into the solution g(x, t) of first approximation, a solution $g + \delta g$ of second approximation and linearizing with respect to δg, we obtain

$$g'''_{x^3} - \alpha_i^2 \frac{U_0}{\nu x_0} g'_x \left(1 - \frac{\Delta x}{x_0} + \dots\right) + \delta g'''_{x^3} - \alpha_i^2 \frac{U_0}{\nu x_0} \delta g'_x = 0,$$

where

$$g'_x = \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \left(g_1 e^{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Delta x}\right) - g_2 e^{-\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Delta x} \quad \text{(first approximation)}.$$

Let us repeat the reasoning used previously, by putting

$$\delta g'_x = b' + 2c' \Delta x + 3e \Delta x^2.$$

The condition in $\delta g'_x$, $\delta g'''_{x^3}$ leads to

$$-\alpha_i^2 \frac{U_0}{\nu x_0} \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \left[g_1 - g_2 + \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \Delta x \,(g_1 + g_2)\right] \Big\} - \frac{\Delta x}{x_0} + \frac{\Delta x^2}{2 x_0^2} \dots \Big\} + 6c$$

$$-\left(\alpha_i^2 \frac{U_0}{\nu x_0}\right)(b' + 2c'\Delta x) + 3e\Delta x^2 = 0.$$

By identification to powers of Δx, we obtain /176

$$6e - \alpha_i^2 \frac{U_0}{\nu x_0} b' = 0 \qquad \text{or} \qquad b' = \frac{6e}{\alpha_i^2 \frac{U_0}{\nu x_0}} = \frac{2}{\alpha_i^2 \frac{U_0}{\nu x_0}} \cdot 3e,$$

$$-\alpha_i^2 \frac{U_0}{\nu x_0} \cdot 2c' + \frac{1}{x_0}\left(\alpha_i^2 \frac{U_0}{\nu x_0}\right)^{\frac{3}{2}} (g_1 - g_2) = 0 \qquad \text{or} \qquad 2c' = \frac{1}{x_0}\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \cdot (g_1 - g_2),$$

$$-\alpha_i^2 \frac{U_0}{\nu x_0} 3e - \left(\alpha_i^2 \frac{U_0}{\nu x_0}\right)^{\frac{3}{2}} \left[\frac{1}{2x_0^2}(g_1 - g_2) - \frac{1}{x_0}\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}(g_1 + g_2)\right] = 0$$

or

$$3e = -\frac{1}{x_0}\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}\left[\frac{g_1 - g_2}{2x_0} - \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}(g_1 + g_2)\right].$$

The complete solution in g'_x becomes

$$g'_x = (g'_x + \delta g'_x) = \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}\left(g_1 e^{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}\Delta x} - g_2 e^{-\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}\Delta x}\right)$$

$$-\frac{2}{x_0\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}}\left[\frac{g_1 - g_2}{2x_0} - \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}(g_1 + g_2)\right] + \ldots$$

and

$$g = g_1 e^{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}\Delta x} + g_2 e^{-\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}} - \frac{2}{x_0\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}}$$

$$\times\left\{\frac{g_1 - g_2}{2x_0} - \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}(g_1 + g_2)\right\}\Delta x + \ldots + g_3$$

$$\cong g_1 e^{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}\left[1 + \frac{2}{x_0 \alpha_i^2 \frac{U_0}{\nu x_0}}\left\{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} - \frac{1}{2x_0}\right\}\right]\Delta x}$$

$$+ g_2 e^{-\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}\left[1 - \frac{2}{x_0 \alpha_i^2 \frac{U_0}{\nu x_0}}\left\{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} + \frac{1}{2x_0}\right\}\right]\Delta x} + g_3$$

of the form of

$$g = g_1 e^{(\lambda + \mu)\Delta x} + g_2 e^{-(\lambda - \mu)\Delta x} + g_3.$$

where g_1, g_2, g_3 are functions of x_0, t_0, and Δt. Then, λ and μ are written as /177

$$\lambda = \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \left[1 - \frac{2}{2 x_0^2 \alpha_i^2 \frac{U_0}{\nu x_0}}\right] = \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}} \left[1 - \frac{\nu}{U_0} \cdot \frac{1}{\alpha_i^2 x_0}\right] \cong \sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}},$$

$$\mu = \frac{2}{x_0} \cdot \frac{1}{\sqrt{\alpha_i^2 \frac{U_0}{\nu x_0}}}.$$

The time equation in g reads

$$\frac{\partial}{\partial t} g''_{x^2} + U_0 g'''_{x^3} - \nu g''''_{x^4} = 0.$$

For the term in $e^{(\lambda+\mu)\Delta x}$, this yields

$$g'_{1_t} (\lambda + \mu)^2 + U_0 g_1 (\lambda + \mu)^3 - \nu g_1 (\lambda + \mu^4) = 0,$$

namely

$$g'_{1_t} + U_0 (\lambda + \mu) \left[1 - \frac{\nu}{U_0} (\lambda + \mu)\right] g_1 = 0,$$

whence

$$g_1 = g_{1_0} e^{-U_0(\lambda+\pi)\left[1 - \frac{\nu}{U_0}(\lambda+\mu)\right]\Delta t} \cong g_{1_0} e^{-U_0(\lambda+\mu)\Delta t}.$$

Similarly,

$$g_2 = g_{2_0} e^{U_0(\lambda-\mu)\left[1 - \frac{\nu}{U_0}(\lambda-\mu)\right]\Delta t} \cong g_{2_0} e^{U_0(\lambda-\mu)\Delta t}.$$

Finally, $g_3 = g_{3_0}$ is constant so that

$$g'_x \cong (\lambda + \mu) g_{1_0} e^{(\lambda+\mu)(\Delta x - U_0\Delta t)} - (\lambda - \mu) g_{2_0} e^{-(\lambda-\mu)(\Delta x - U_0\Delta t)} = e^{\mu(\Delta x - U_0\Delta t)}$$
$$\times \left[\lambda \left\{ g_{1_0} e^{\lambda(\Delta x - U_0\Delta t)} - g_{2_0} e^{-\lambda(\Delta x - U_0\Delta t)} \right\} + \mu \left\{ g_{1_0} e^{\lambda(\Delta x - U_0\Delta t)} + g_{2_0} e^{-\lambda(\Delta x - U_0\Delta t)} \right\}\right].$$

The boundary conditions with respect to y are always $u'(0) = 0$ (in general) and $v'(0) = 0$, with the first being directly satisfied by putting $\varphi_1 = \varphi_2 = \frac{1}{2}$. Let us also set

$$\Delta X = \Delta x - U_0 \Delta t.$$

$$(\varphi f'_x + g'_x)_{y=0} = f'_x + g'_x = \left[\beta \left(f_{1_0} e^{\beta\Delta x} - f_{2_0} e^{-\beta\Delta x}\right) + \gamma \left(f_{1_0} e^{\beta\Delta x} + f_{2_0} e^{-\beta\Delta x}\right)\right] e^{\gamma\Delta x}$$
$$+ \left[\lambda \left(g_{1_0} e^{\lambda\Delta x} - g_{2_0} e^{-\lambda\Delta x}\right) + \mu \left(g_{1_0} e^{\lambda\Delta x} + g_{2_0} e^{-\lambda\Delta x}\right)\right] e^{\mu\Delta x}$$
$$\cong \left[\beta (f_{1_0} - f_{2_0}) + \lambda (g_{1_0} - g_{2_0}) + \gamma (f_{1_0} + f_{2_0}) + \mu (g_1 + g_{2_0})\right]$$

$$+ \Delta X [\gamma \} \beta (f_{1_0} - f_{2_0}) + \gamma (f_{1_0} + f_{2_0})\}$$
$$+ \mu \{ \lambda (g_{1_0} - g_{2_0}) + \mu (g_{1_0} + g_{2_0}) \} + \beta \} \beta (f_{1_0} + f_{2_0}) + \gamma (f_{1_0} - f_{2_0}) \}$$
$$+ \lambda \{ \lambda (g_{1_0} + g_{2_0}) + \mu (g_{1_0} - g_{2_0})] \dots = 0.$$

Each of the expressions in brackets must be canceled, which necessitates two conditions to be written in each x_0 for which, in principle, four constants f_{1_0}, f_{2_0}, g_{1_0}, g_{2_0} are available. /178

However, it is necessary here to allow for the fact that β is imaginary. For $\beta = \mathring{i}\bar{\beta}$, a cancellation of the expressions in brackets yields

$$\mathfrak{R}\mathcal{J}: \ \bar{\beta} (f_{1_0} - f_{2_0}) = 0 \text{ and } \lambda (g_{1_0} - g_2) + \gamma (f_{1_0} + f_{2_0}) + \mu (g_{1_0} + g_{2_0}) = 0;$$
$$\mathfrak{R}\mathcal{R}: \ \bar{\beta} [2 \gamma (f_{1_0} - f_{2_0})] = 0$$

and

$$(-\beta^2 + \gamma^2)(f_{1_0} + f_{2_0}) + 2 \mu \lambda (g_{1_0} + g_{2_0}) + (\mu^2 + \lambda^2)(g_{1_0} - g_{2_0}) = 0.$$

It follows that $f_{1_0} = f_{2_0}$, so that only two conditions remain that determine g_{1_0} and g_{2_0} as a function of x_0 and f_{1_0} which remains always available* for satisfying a complementary condition.

Since the only role assigned to the function g(x) is to permit satisfying the condition at the wall v(0) = 0, its further calculation is unnecessary here.

For x_c corresponding to usual values of Reynolds numbers of the end of transition ($\mathfrak{R}_c > 0.5 \times 10^6$, for example), k will be large and will be expressed in 10^3. The second approximation modification

$$\delta f'_x = b = -\frac{2}{x} \cdot \frac{\alpha_i^2 \frac{U_0}{\nu x}}{k^2 - \alpha_i^2 \frac{U_0}{\nu x}}$$

will then be small with respect to the corresponding function of first approximation:

$$f'_x = -\left(k^2 - \alpha_i^2 \frac{U_0}{\nu x}\right)(f_1 + f_2),$$

* Here, f_{1_0} and f_{2_0} will be complex quantities in general, except if the arbitrary origin of time had been so selected as to keep these quantities real, as we already demonstrated above.

since, if $\bar{x}$ represents the value of x canceling $k^2 - \alpha_i^2 \frac{U_0}{\nu x}$, we obtain

$$\frac{\delta f'_x}{f'_x} = \frac{\frac{2}{x} \cdot \alpha_i^2 \frac{U_0}{\nu x}}{\left(k^2 - \alpha_i^2 \frac{U_0}{\nu x}\right)^2} = \frac{2}{x k^2} \frac{\frac{\bar{x}}{x}}{\left(1 - \frac{\bar{x}}{x}\right)^2}.$$

Except in the domain $x \to \bar{x}$, this simultaneously justifies the expansions of second approximation carried out above as well as the possibility of limiting the calculations to those of the first approximation.

For x close to $\bar{x}$, it is necessary to return to the fundamental equation /179

$$f'''_{x^3} + f'_x \left(k^2 - \alpha_i^2 \frac{U_0}{\nu x}\right) = 0,$$

which reduces to

$$f'''_{x^3} \cong 0.$$

Its solution reads

$$f'_x = \bar{f}_1 (x - \bar{x}), \qquad f = f_1 \frac{(x - \bar{x})^2}{2}$$

(where the propagation starts in such a manner that, in $x = \bar{x}$, we have $f = 0$, $f'_x = 0$). This eliminates the singularity about $\bar{x}$.

APPENDIX IV

VELOCITY DISTRIBUTION IN THE BOUNDARY SUBLAYER IN INCOMPRESSIBLE REGIME

The boundary conditions, characterizing the boundary sublayer ϵ - as will be recalled - are as follows:

at the wall $y_w = 0$,

$$U_w = 0, \quad V_w = 0;$$

at the interface with the actual boundary layer, $y = \epsilon$,

$$U = U_1 \quad \text{where} \quad U_1 \to X \cong 0.45\, U_0,$$

constant for the asymptotic state;
local (asymptotic) friction coefficient, being

$$\overline{C}_f^* = \frac{\tau}{\frac{\rho}{2} U_0^2} = \frac{\mu}{\frac{\rho}{2} U_0^2}\left(\frac{\partial U}{\partial y}\right)_w = \frac{2\nu}{U_0^2}\left(\frac{\partial U}{\partial y}\right)_w.$$

This coefficient must correspond to that calculated by the equation of loss of momentum:

$$\overline{C}_f^* \cong 0.75 \sqrt[4]{\frac{\overline{\beta}}{\mathcal{R}}} \quad \text{where} \quad \overline{\beta} \cong 0.7 \cdot 10^2, \quad \mathcal{R} = \frac{U_0 x}{\nu}.$$

Let us return to the Navier-Stokes equations

$$\frac{1}{\rho} \cdot \frac{\partial p}{\partial x} = -\left[U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right] + \nu\left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right],$$

$$\frac{1}{\rho} \cdot \frac{\partial p}{\partial y} = -\left[U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right] + \nu\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right]$$

and to the continuity condition

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0.$$

Hence, /181

$$V(y) = -\int_0^y \frac{\partial U}{\partial x}\, dy \quad (\text{since} \quad V_w = 0).$$

It follows from this that

$$V(\epsilon) = -\int_0^\epsilon \frac{\partial U}{\partial x}\, dy$$

where $\frac{\partial U}{\partial x}$ is of the second order of smallness and ϵ is very small. Thus, we have $V(\epsilon) \cong 0$ and $V(y) \cong 0$ for $0 < y \leq \epsilon$.

With $\frac{\nu}{U_0 x}$ which itself is very small, the Navier-Stokes equations in the sublayer ϵ reduce to

$$\frac{1}{\rho}\cdot\frac{\partial p}{\partial x} \cong -U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2},$$
$$\frac{1}{\rho}\cdot\frac{\partial p}{\partial y} \cong 0.$$

Since the gradient p'_y is zero in the sublayer, the gradient p'_x will depend only on that existing along the interface U_1 with the actual boundary layer, for which

$$V \cong 0, \qquad U = (U_1 + u)_{y \to 0}.$$

Hence,

$$\left(\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}\right)_\epsilon \cong -U_1\frac{\partial U_1}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y \to 0}.$$

In the symptotic state, $U_1 \to X$ is constant and $\frac{\partial U_1}{\partial x} = 0$.

With our conventions, $\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y \to 0}$ is of an order higher than 2 and thus can be neglected. Consequently, $\left(\frac{\partial p}{\partial x}\right)_{y \leq \epsilon} \cong 0$.

This places us into the frame of the Prandtl-Blasius approximations and hypotheses for which the conditions originating from Navier-Stokes are written as follows:

$$U\frac{\partial u}{\partial x} - \nu\frac{\partial^2 U}{\partial y^2} = 0.$$

The sublayer is laminar (in the incompressible regime).

Let us put, as is conventional,

$$\eta = y\sqrt{\frac{U_0}{\nu x}} = y\,\frac{U_0}{\nu}\sqrt{\frac{1}{\mathcal{R}}},$$
$$\psi = \sqrt{\nu x U_0}\; f(\eta)$$
$$\psi'_y = U, \qquad \psi'_x = -V.$$

Hence, /182

$$U = U_0 f'_\eta, \qquad V = \frac{1}{2x}\sqrt{\nu x U_0}\,[\eta f'_\eta - f],$$

with, since $V \cong 0$,

$$\eta f'_\eta \cong f.$$

Finally,

$$U'_x = -U_0 f''_{\eta^2} \cdot \frac{\eta}{2x}, \quad U''_{y^2} = U_0 f'''_{\eta^3} \frac{U_0}{\nu x}.$$

The Navier-Stokes condition becomes

$$\eta f'_\eta f''_{\eta^2} + 2 f'''_{\eta^3} = 0.$$

Since $\eta f'_\eta \cong f$, the Blasius equation of the second order is obtained:

$$f f'_{\eta^2} + 2 f'''_{\eta^3} \cong 0.$$

It is known that the calculation procedure for $f(\eta)$ consists in putting

$$f(\eta) = A_0 + A_1 \eta + \frac{A_2}{2!}\eta^2 + \ldots + \frac{A_n}{n!}\eta^n + \ldots$$

and then in calculating f'_{η^2}, f'''_{η^3} so as to write the Blasius relation in the form of a polynomial to powers of η. Each of the factors must be zero so that we obtain

$$A_0 = A_1 = 0, \qquad A_3 = A_4 = 0, \qquad A_6 = A_7 = 0, \qquad A_9 = A_{10} = 0, \ldots$$

$$A_2 \neq 0, \qquad A_5 = -\frac{A_2^2}{2}, \qquad A_8 = 2.5\, A_2^3 \ldots,$$

whence

$$U = U_0 A_2 \left[\eta - \frac{A_2}{2 \cdot 4!}\eta^4 + \ldots\right], \qquad \frac{\partial U}{\partial Y} = U_0 \sqrt{\frac{U_0}{\nu x}}\, A_2 \left[1 - \frac{A_2}{2 \cdot 3!}\eta^3 + \ldots\right].$$

Let us substitute these expressions into the conditions given at the beginning of this Appendix, concerning the characteristics of the sublayer, namely:

condition of the velocity gradient at the wall:

$$\frac{2\nu}{U_0^2} U_0 \sqrt{\frac{U_0}{\nu x}}\, A_2 = \bar{C}_f = 0.75 \sqrt[4]{\frac{\beta}{\mathcal{R}}},$$

i.e.,

$$A_2 = \sqrt{\frac{U_0 x}{\nu}}\, 0.375 \sqrt[4]{\frac{\beta}{\mathcal{R}}} = 0.375 \sqrt[4]{\beta \mathcal{R}} \cong 0.061 \sqrt[4]{\mathcal{R}};$$

connectivity condition of velocity at the boundary ε:

$$0.45\ U_0 = U_0\ A_2 \left[\eta_\varepsilon - \frac{A_2}{2 \cdot 4!}\ \eta_\varepsilon^4 \cdots \right],$$

i.e., /183

$$0.45 = 0.375 \sqrt[4]{\beta\ \mathcal{R}} \left[\eta_\varepsilon - \frac{0.375 \sqrt[4]{\beta\ \mathcal{R}}}{48} \cdot \eta_\varepsilon^4 \cdots \right].$$

It is easy to demonstrate that the term $\frac{0.375}{48} \sqrt[4]{\beta \mathfrak{R}} \cdot \eta^3$ is extremely small with respect to unity, no matter what the value of $\mathfrak{R} > 10^3$ might be.

Hence,

$$\eta_\varepsilon \cong \frac{8.45}{0.375 \sqrt[4]{\beta\ \mathcal{R}}} \cong \frac{1}{0.136 \sqrt[4]{\mathcal{R}}}$$

and

$$U \cong U_0\ A_2\ \eta = U_0 \cdot 0.061 \sqrt[4]{\mathcal{R}}\ \frac{U_0}{\nu} \sqrt{\frac{1}{\mathcal{R}}} \cdot Y,$$

i.e.,

$$\frac{U(Y)}{U_0} \doteq \frac{U_0}{\nu} \cdot 0.061 \sqrt[4]{\mathcal{R}} \cdot Y \qquad \text{for} \qquad 0 < Y \leqslant \varepsilon.$$

Thus U(Y) is quasi-linear in Y in the sublayer which, in the incompressible regime, is laminar (Q.E.D.).

At the interface with the actual boundary layer a discontinuity of the velocity gradient would then exist. However, this results from the approximations made, specifically from the study of the actual boundary layer where the viscous terms $\nu \frac{\partial^2 U}{\partial y^2}$ have been neglected (in the problem in y), which is valid for $Y >> \varepsilon$ but is not strictly valid for $Y \cong \varepsilon$. Thus, this discontinuity results from a local failure of the calculation but does not imply any contradiction (with respect to this subject, see Chapt.II, Sect.8).

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